Cayley-Klein geometries: a modern historical perspective

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• Tesina:
  ✤ Contracciones ultrarelativistas del grupo de Poincaré, 1981

• I. M. Yaglom
**The Nine Geometries of Cayley-Klein type**

- **The nine two-dimensional CK spaces** 
  \[ S^2_{[\kappa_1,\kappa_2]} = SO_{\kappa_1,\kappa_2}(3)/SO_{\kappa_2}(2). \]
  - ✔ Singularized by two real parameters \( \kappa_1, \kappa_2 \)
  - ★ Values \( \kappa_1 > 0, = 0, < 0 \) denoted +, 0, − label columns; similarly \( \kappa_2 \) vs. rows.

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- **All these geometries appear realized in Nature**
The many faces of $\kappa_1, \kappa_2$ [1]

- **Projective view**
  - $\kappa_1 >, =, < 0 \equiv$ Elliptic, Parabolic, Hyperbolic type of measure of **distances**
  - $\kappa_2 >, =, < 0 \equiv$ Elliptic, Parabolic, Hyperbolic type of measure of **angles**

- **Old synthetic view**
  - $\kappa_1 >, =, < 0 \equiv$ a number $0, 1, \infty$ of lines through a given point $P$ and not meeting a given line $l$ (not through $P$).
  - $\kappa_2 >, =, < 0 \equiv$ a number $0, 1, \infty$ of points on a given actual line $l$ and not joined to a given point $P$ by an actual line ($P$ not on $l$).

- **Differential Geometry view**
  - $\kappa_1 >, =, < 0 \equiv$ Positive, Zero, Negative **constant curvature** $\kappa_1$
  - $\kappa_2 >, =, < 0 \equiv$ Positive Definite, Degenerate, Lorentzian, metric reducible to $\text{diag}\{+1, \kappa_2\}$ at each point: **signature** $\kappa_2$.

- **Limiting view**: Cases where either $\kappa_1, \kappa_2$ are zero are limiting approximations to the generic cases where both $\kappa_1, \kappa_2$ are different from zero
  - $\kappa_1 \to 0$ limit around a point
  - $\kappa_2 \to 0$ limit around a line
Two separate families? Riemannian . . . and Lorentzian

**Riemannian spaces** Riemann’s far-reaching extension of the Euclidean space $E^n$

- **Two steps** Zero Curvature $\rightarrow$ Constant Curvature $\rightarrow$ General Curvature.

**Constant curvature Riemannian spaces**

- Essentially, a one-parameter family of n-d Riemannian spaces of constant curvature grouped in three types $S^n_\kappa, E^n, H^n_\kappa$ according as $\kappa > 0, = 0, < 0$.
- Standard choice $\kappa = 1, 0, -1$ gives the three standard $S^n, E^n, H^n$.

**Lorentzian spaces** A (similar) extension of the Lorentz-Minkowski space $M^{1,n}$

- No essential changes from Riemannian

**Constant Curvature PseudoRiemannian (Lorentzian) spaces**

- Essentially, a one-parameter family of (1+n)-d Lorentzian spaces of constant curvature $\text{AdS}^{1+n}_\kappa, M^{1+n}, dS^{1+n}_k$ according as $\kappa > 0, = 0, < 0$.
- Standard choice $\kappa = 1, 0, -1$ gives the three standard $\text{AdS}^{1+n}_\kappa, M^{1+n}, dS^{1+n}_k$. 
CK Spaces Versus CK Geometries

• The spaces discussed so far can be seen under the Riemannian and Kleinian perspectives

• Simplest example: Ordinary Euclidean space $E^2$ is a symmetric homogeneous space of the Euclidean group $ISO(2)$, $E^2 \approx ISO(2)/SO(2)$.

  ✓ Elements of this space are the points in Euclidean geometry, and the involution $\Pi_1$ correspond to reflection in a point.

• Yet there is another symmetric homogeneous space in Euclidean Geometry: The set of all lines in $E^2$. This is $\tilde{E}^2 \approx ISO(2)/ISO(1)$.

  ✓ Elements of this space are the lines in Euclidean geometry, and the involution $\Pi_2$ correspond to reflection in a line.

• (Symmetric) Geometry: An interlinked set of homogeneous spaces associated to the same group $G$ but with a set of commuting involutive automorphisms.
Symmetric homogeneous spaces of ‘Cayley-Klein type’ [1]

- I discuss only the real 2d case, everything works for the real, Hermitian complex and quaternionic spaces, in any $n$.

- **Look for 3d Lie groups $G$** which allow for two commuting involutive automorphisms in the corresponding Lie algebra.
  - These would provide two symmetric homogeneous spaces of the Lie group $G$

- **Approach** Look in the common eigenbasis $\{P_1, P_2, J\}$ The more general such (quasi-simple) Lie algebra having $\Pi_1, \Pi_2$ as automorphisms depends on two real parameters $\kappa_1, \kappa_2$

  \[
  \Pi_1 : (P_1, P_2, J) \rightarrow (-P_1, -P_2, J), \quad \Pi_2 : (P_1, P_2, J) \rightarrow (P_1, -P_2, -J)
  \]

  \[
  [P_1, P_2] = \kappa_1 J \quad [J, P_1] = P_2 \quad [J, P_2] = -\kappa_2 P_1
  \]

- Denote $SO_{\kappa_1, \kappa_2}(3)$ the Lie groups obtained by exponentiation
  - **One-parameter subgroup invariant under involution** $\Pi_1$ generated by $J$:

  \[
  \exp(\gamma J) = \begin{pmatrix}
  1 & 0 & 0 \\
  0 & C_{\kappa_2}(\gamma) & -\kappa_2 S_{\kappa_2}(\gamma) \\
  0 & S_{\kappa_2}(\gamma) & C_{\kappa_2}(\gamma)
  \end{pmatrix}
  \]

  $SO_{\kappa_2}(2)$
Labeled Trigonometric functions: Labelled ‘cosine’ $C_\kappa(x)$ and ‘sine’ $S_\kappa(x)$:

\[
C_\kappa(x) := \begin{cases} 
\cos \sqrt{\kappa} x \\
1 \\
\cosh \sqrt{-\kappa} x 
\end{cases} \quad S_\kappa(x) := \begin{cases} 
\frac{1}{\sqrt{\kappa}} \sin \sqrt{\kappa} x & \kappa > 0 \\
x & \kappa = 0 \\
\frac{1}{\sqrt{-\kappa}} \sinh \sqrt{-\kappa} x & \kappa < 0
\end{cases}
\]

Deformations of the two basic functions $1$ and $x$

Natural realization of the CK group $SO_{\kappa_1,\kappa_2}(3)$ as a group of linear transformations in an ambient linear space $R^3 = (x^0, x^1, x^2)$.

Therefore $SO_{\kappa_1,\kappa_2}(3)$ acts in $R^3$ as linear isometries of a bilinear form with $\Lambda_{\kappa_1,\kappa_2} = \text{diag}\{+1, \kappa_1, \kappa_1 \kappa_2\}$ as metric matrix.

CK space as Homogeneous symmetric space: $S^2_{[\kappa_1],\kappa_2} \equiv SO_{\kappa_1,\kappa_2}(3)/SO_{\kappa_2}(2)$

Natural structures in these homogeneous symmetric spaces:

- A canonical connection (compatible with the metric).
- A metric coming from the Killing-Cartan form in the Lie algebra. The metric is of constant curvature $\kappa_1$ and of ‘signature type’ $\kappa_2$.

Hence this family includes precisely the spaces of constant curvature (either $> 0$, $-0$, $< 0$) and (quadratic) metric of either signature type.
### The nine CK 2d spaces as ‘spheres’ in ambient space coordinates

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★ **Weierstrass ambient description as ‘CK spheres’**

\[
(x^0)^2 + \kappa_1 (x^1)^2 + \kappa_1 \kappa_2 (x^2)^2 = 1
\]

\[
\kappa_1 = 1, 0, -1 \quad \text{(columns, left to right)} \quad \kappa_2 = 1, 0, -1 \quad \text{(rows, up to down)}
\]

★ **Metric in the ambient space**

\[
dl^2 = (dx^0)^2 + \kappa_1 (dx^1)^2 + \kappa_1 \kappa_2 (dx^2)^2.
\]

✓ Metric in the CK space $ds^2 := \frac{1}{\kappa_1} dl^2$
The nine CK 2d spaces as ‘spheres’ in ambient space coordinates
The three $S^2$, $E^2$, $H^2$ CK 2d spaces

✓ For distances $r$ and angles $\theta$, business as usual
✓ In Minkowski space, rapidities appear through hyperbolic trig functions. $\kappa^2 < 0$ is a negative (hyperbolic) label. Standard choice $\kappa^2 = -1$

✓ But real rapidities (‘angles’) do not cover the full Minkowski, AntiDeSitter or DeSitter spaces

✓ Introduce a ‘quadrant’ with negative label $\kappa = -1$ defined so that $\cdot_{-1}$ is the ‘rapidity’ between orthogonal vectors in this space. $\cdot_{-1} = \frac{\pi}{2i}$

✓ Allow values $\chi, \chi + \cdot_{-1}, \chi + 2\cdot_{-1}, \chi + 3\cdot_{-1}$ for the rapidity

✓ Now rapidities cover the full Minkowski space (and $AdS$ and $dS$ too)
Something new in the three $AdS^2$, $M^2$, $dS^2$ spaces? II

- In Minkowski space ($\kappa_1 = 0$), proper times (the ‘distances’, denoted $r$) appear through parabolic trig functions
  
  ✓ These real ‘distances’ do not cover the full Minkowski $M^2$
  ✓ The basic metric is a quadratic form which is not definite positive
  ✓ Introduce ‘ideal’ distances and allow the ‘distances’ to be either real $r$ or pure imaginary $ir$

  ![Diagram](image.png)

  ✓ Now these ‘distances’ jointly with the extended ‘angles’ cover all $M^2$
  ✓ How about to extend this idea to all CK spaces?
Labeled Trigonometric functions: Recall ‘cosine’ $C_\kappa(x)$ and ‘sine’ $S_\kappa(x)$ with ‘label’ $\kappa$ are defined initially for real variable $x$ (later for some particular complex values of $x$) as:

$$C_\kappa(x) := \begin{cases} \cos \sqrt{\kappa} x \\ 1 \\ \cosh \sqrt{-\kappa} x \end{cases} \quad S_\kappa(x) := \begin{cases} \frac{1}{\sqrt{\kappa}} \sin \sqrt{\kappa} x & \kappa > 0 \\ x & \kappa = 0 \\ \frac{1}{\sqrt{-\kappa}} \sinh \sqrt{-\kappa} x & \kappa < 0 \end{cases}.$$ 

Define a quadrant: $\downarrow \kappa = \frac{\pi}{2\sqrt{\kappa}}$.

Expression for the ambient space coordinates in terms of ‘naive’ polar coordinates

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} C_{\kappa_1}(r) \\ S_{\kappa_1}(r) C_{\kappa_2}(\theta) \\ S_{\kappa_1}(r) S_{\kappa_2}(\theta) \end{pmatrix}.$$ 

but, what is the domain of the coordinates $r, \theta$?

For an hyperbolic quantity, (e.g, the minkowskian rapidity angle $\theta$), $\downarrow -1$ is a pure imaginary quantity. Should this mean that we have to accept any complex argument in the sine and cosine functions?
The full CK domain for the CK trigonometric functions

✓ No!. The natural requirement is to enforce that the squares of $S_\kappa(x)$ and $C_\kappa(x)$ should be real.

✓ This determines a subset of the complex plane, which is a kind of ‘branched one dimensional set’. This is called the full domain of the CK variable with label $\kappa$.

✓ This the the full CK domain with label $\kappa = -1$

✓ Branching points at 0, $\pm \sqrt{-1}$, $\pm 2\sqrt{-1} \equiv 0_{-1}$ and as $\pm \infty$. 

\[ \begin{align*}
\end{align*} \]
The full CK domain of a variable with positive label $\kappa$. 
The full CK domain of a variable with any label $\kappa$

✓ The CK domain of a CK variable, for any value of $\kappa$ is the set of the following values (here $x, y$ are real, and $0 \equiv 2 \cdot \cdot$)

✓ Actual and antiactual values, $x, 2 \cdot \cdot + x$ (depicted in deep blue)

✓ Coactual and anticoactual values, $\pm \cdot \cdot + x$ (depicted in cyan)

✓ Ideal and antiideal values, $iy, 2 \cdot \cdot + iy$ (depicted in red and orange)

✓ Coideal and anticoideal values, $\pm \cdot \cdot + iy$ (depicted in magenta)

✓ Essential fact: The domain of a variable depends on its label
The stereocentral (stereognomonic) model of the nine CK spaces

- **A new ‘projection’ to display the nine spaces at once**
  - Essentially, extends the visually ‘good’ traits of the stereographic projection in the $S^2$
The stereocentral model of the nine CK spaces

✓ For any CK space, geodesics are represented as ‘affine’ circles cutting the equator antipodally
✓ The ‘North’ and ‘South’ hemispaces are represented as the interior and exterior of the ‘Equator’ circle
The basic coordinate: distance to a point

- **Fix the point** \( O \) **at the origin**
  - For any other point \( P \) (in the ‘full’ CHK space) passes a unique geodesic linking \( P \) **to** \( O \)
    - Antipodal exception
  - Define \( r \) the coordinate \( r \) as the ‘extended’ parameter of separation along this geodesic

- **This** \( r \) **is defined in the ‘full’ CK space**
  - \( r \) could be either actual, coactual or ideal, coideal, or its anti versions
The coordinate lines $r = \text{cte}$ are circles with center at $O_0$ in the geometry of each CK space.
The coordinate lines $\theta = \text{cte}$ are geodesics through $O_0$ in the geometry of each CK space.
The duality in the CK scheme [1]

- **Duality is an interchange between the basic elements in the CK original space and the ones in the dual, according to:**

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<th><em>versus</em></th>
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★ **The map** $\mathcal{D}$ leaves the Lie algebra invariant, interchanges the two constants $\kappa_1 \leftrightarrow \kappa_2$, and hence the space of points with the space of (actual) lines, $S^2_{[\kappa_1],[\kappa_2]} \leftrightarrow S^2_{[\kappa_2],[\kappa_1]}$.

★ **In the sphere** $S^2$ this is the well known polarity.

★ **Duality relates two CK geometries which are different in general.** Only when $\kappa_1 = \kappa_2$ the CK geometry is self-dual. Examples: $S^2, G^{1+1}, dS^{1+1}$.

★ **Theorem** The dual of a CK space with curvature $\kappa_1$ and metric of signature type $(+, \kappa_2)$ is the CK space with curvature $\kappa_2$ and metric of signature type $(+, \kappa_1)$.
The duality in the CK scheme [3]

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- **Duality is realized by a symmetry along the main diagonal.**

\[ \mathcal{D} : \begin{array}{c}
S^2_{(1,1)} \leftrightarrow S^2_{(1,1)} \\
H^2_{(-1,1)} \leftrightarrow AdS^{1+1}_{(1,-1)} \\
dS^{1+1}_{(-1,-1)} \leftrightarrow dS^{1+1}_{(-1,-1)}
\end{array} \]

- The sphere \( S^2 \) and the DeSitter space \( dS^{1+1}_- \) are autodual.
- Hyperbolic plane \( H^2 \) and the AntiDeSitter space \( AdS^{1+1}_+ \) are mutually dual.
Visualizing duality in the stereocentral model: The dual of $H^2$ is $AdS^{1+1}$

- All the lines orthogonal to $l_1$ meet in a single point.
- This point is in the Ideal sector of $H^2$, which is $AdS^{1+1}$.
- Lines orthogonal to $l_1$ and the complete system of associated orthogonal coordinate net, covering the Actual and Ideal Sectors of $H^2$. 
Some applications: The classification of the confocal coordinate systems

- **Generic systems**
  - ✓ Elliptic (actual foci, actual focal separation)
  - ✓ Parabolic (one foci actual, other focus coactual, coactual focal separation)
  - ✓ CoElliptic (coactual foci, actual focal separation)

- **Limiting systems (non generic)**

- **Particular systems, for special values of the focal separation, e.g.**
  - ✓ Equiparabolic systems, with focal separation equal to a quadrant
  - ✓ Isoelliptic (two focus with isotropic separation)
  - ✓ ...

  * Horosystems
  - ✓ HoroElliptic (one actual focus, other focus at infinity)
  - ✓ HoroCoElliptic (one coactual focus, other focus at infinity)

  * Coalescing foci
  - ✓ Polar, parallel, horocyclic
A sample


Complete description of the central extensions of all Lie algebras \( so_{\kappa_1,\ldots,\kappa_N} (N+1) \), \( su_{\kappa_1,\ldots,\kappa_N} (N+1) \) and \( sp_{\kappa_1,\ldots,\kappa_N} (N+1) \) in the three ‘real, complex and quaternionic type’ Cayley Klein series, for any \( N \) and general \( \kappa_i \).

An exhaustive and complete study of trigonometry in all rank-one spaces of real and ‘complex’ type, both made in a completely general CK fashion, with \( \kappa_1, \kappa_2 \) and \( \eta \) as parameters. Real spaces are related to space-time (they include all homogeneous models of non-relativistic and relativistic space-times), and complex ‘hermitian’ ones include the quantum space of states.

Several papers where the CK \( \kappa_1, \kappa_2 \) scheme is used in relation to quantum deformations of the classical CK groups and algebras.
Beyond

- **The scheme encompasses all symmetric homogeneous spaces**
  - Real, complex, quaternionic type symmetric homogeneous spaces in any dimension, and exceptional ones as well (connection with octonions).
  - For instance, in the complex hermitian case (the CK general version of $su(N)$), it turns out that there are $n$ commuting involutions). The extra involution appear as a Cayley-Dickson parameter, leading to spaces over the three composition algebras of complex numbers, Study numbers and split complex numbers.

- **Dynamics, Integrability and superintegrability in CK spaces**
Congratulations, Miguel Angel

General Properties or propositions should be more easily demonstrable than any special case of it

- **J. I. Sylvester**  Note on Spherical Harmonics, *Phil. Mag.* (1876)

Thank you very much,
Any comment, criticism, reference, . . . , welcome at **msn@fta.uva.es**