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# Stochastic resonance in overdamped systems with fractional power nonlinearity

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**Abstract.** The stochastic resonance phenomenon in overdamped systems with fractional power nonlinearity is thoroughly investigated. The first kind of nonlinearity is a general fractional power function. The second kind of nonlinearity is a fractional power function with deflection. For the first case, the response is clearly divergent for some fractional exponent values. The curve of the spectral amplification factor *versus* the fractional exponent presents some discrete regions. For the second case, the response will not be divergent for any fractional exponent value. The spectral amplification factor decreases with the increase in the fractional exponent. For both cases, the nonlinearity is the necessary ingredient to induce stochastic resonance. However, it is not the sufficient cause to amplify the weak signal. On the one hand, the noise cannot induce stochastic resonance in the corresponding linear system. On the other hand, the spectral amplification factor of the nonlinear system is lower than that of the corresponding linear system. Through the analysis carried out in this paper, we are able to find that the system with fractional deflection nonlinearity is a better stochastic resonance system, especially when an appropriate exponent value is chosen. The results in this paper might have a certain reference value for signal processing problems in relation with the stochastic resonance method.

## 1 Introduction

Stochastic resonance (SR) is a well-known nonlinear phenomenon in a wide range of science and engineering fields. When SR occurs, the weak low-frequency signal in the output can be amplified by an appropriate amount of noise. It was first put forward by Benzi *et al.* [1, 2] in the investigation of the ice ages. Since then, it has been investigated in many different systems, like the ring laser [3, 4], biological systems [5–7], neuronal networks [8–10], image processing [11, 12], signal processing [13, 14], energy harvesting [15–17], fault diagnosis [18–20], etc. In the early stage of SR investigations, the systems considered were basically nonlinear models. With the development of the theory, researchers found that SR not only occurs in nonlinear systems, but it can also occur in some linear systems, when some coupling elements exist in the system [21–23]. Moreover, besides the traditional SR occurs at the excitation frequency, it may also occur at some superharmonic or subharmonic frequencies in the nonlinear response regime [24–27].

Although there are many references on SR, to our knowledge, there is little work on SR in systems with a fractional power nonlinearity. However, the nonlinearity of the system may be in a fractional power function form. Some works discussing the dynamical properties of this kind of systems are found in refs. [28–35]. Hence, in this paper, we will study overdamped systems with typical fractional power nonlinearity.

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The first kind of system is an overdamped system with a general factional power function,

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} - ax(t) + bx^{\alpha}(t) = f\cos(\omega t) + \xi(t), \tag{1}$$

and we will use the term OGF system to refer to it all throughout this paper. In eq. (1), the parameters satisfy the conditions $a > 0$, $b > 0$ and $\alpha > 1$. The term $f\cos(\omega t)$ is a weak low-frequency excitation. $\xi(t)$ is a Gaussian white noise with zero mean value and noise intensity $\sigma$, *i.e.*,

$$\langle \xi(t) \rangle = 0,$$

$$\langle \xi(t)\xi(t') \rangle = 2\sigma\delta(t - t'). \tag{2}$$

In eq. (1), the nonlinearity appears as a general fractional power function. Specifically, the exponent $\alpha$ can be an integer or a fractional number. The unstable equilibrium of system (1) is $x_0 = 0$. For some cases, the stable equilibrium of the system is $x_1 = \sqrt[\alpha-1]{\frac{a}{b}}$. For other cases, the stable equilibria of the system are $x_{1,2} = \pm\sqrt[\alpha-1]{\frac{a}{b}}$. Specifically, the shape of the potential function depends on the value of the fractional exponent.

The second kind of system is an overdamped system with fractional deflection nonlinearity,

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} - ax(t) + bx(t)|x(t)|^{\alpha-1} = f\cos(\omega t) + \xi(t), \tag{3}$$

and we will refer to the ODF system in this paper. In eq. (3), the parameters still satisfy the conditions $a > 0$, $b > 0$ and $\alpha > 1$. The exponent $\alpha$ is also an integer or a fractional number as that in system (1). $f\cos(\omega t)$ and $\xi(t)$ have the same properties as in eq. (1). The unstable equilibrium of system (3) is $x_0 = 0$ and two stable equilibria of the system are given by $x_{1,2} = \pm\sqrt[\alpha-1]{\frac{a}{b}}$. In other words, the ODF system is always bistable. The shape of the potential function is independent of the fractional exponent $\alpha$.

The outline of the paper is organized as follows. In sect. 2, we investigate the SR phenomenon in the OGF system. In sect. 3, we investigate the SR phenomenon in the ODF system. Both in sects. 2 and 3, effects of the noise, the fractional exponent, the nonlinearity on SR are discussed in detail. Finally, the main results of this paper are described in sect. 4.

## 2 The SR phenomenon in the OGF system

There are some different indexes to measure the SR degree. In this paper, we use the spectral amplification factor as the index to evaluate the SR. The spectral amplification factor [25], labelled by $\eta$, is defined by

$$\eta = \left(\frac{\bar{x}}{f}\right)^2, \tag{4}$$

where $\bar{x}$ is the mean value of the response amplitude at the excitation frequency $\omega$. In our work, we use 1000 different noise realizations to simulate the stochastic process. For each value of the response amplitude, $Q$, at the frequency $\omega$, is calculated by

$$Q(\omega) = \sqrt{Q_{\sin}^2(\omega) + Q_{\cos}^2(\omega)}, \tag{5}$$

where $Q_{\sin}(\omega) = \frac{2}{rT}\int_0^{rT} x(t)\sin(\omega t)\mathrm{d}t$, $Q_{\cos}(\omega) = \frac{2}{rT}\int_0^{rT} x(t)\cos(\omega t)\mathrm{d}t$. The parameter $r$ is a positive and integer number, which should be large enough. In this paper, we let $r = 100$. The period of the low-frequency excitation is $T$.

### 2.1 The noise induced SR

In fig. 1, under different values of the fractional exponent and the signal amplitude, some curves of the spectrum amplification *versus* the noise intensity are given. In fig. 1(a), $\alpha = 1.4$, when $f = 0.05$ and $f = 0.1$, there is a slight SR phenomenon induced by the noise intensity. When $f = 0.3$, the spectrum amplification factor decreases with the increase of the noise intensity. In other words, there is no SR phenomenon for stronger low-frequency signal. In figs. 1(b) and (f), $\alpha = 2$ and $\alpha = 4$, respectively. In these two cases, the response of the system will be rapidly divergent with the increase of the noise intensity. Moreover, in all cases in figs. 1(a), (b), (c), (e), (f), (g), the potential function has only one stable equilibrium and one unstable equilibrium. It makes the response of the system to be divergent easily.

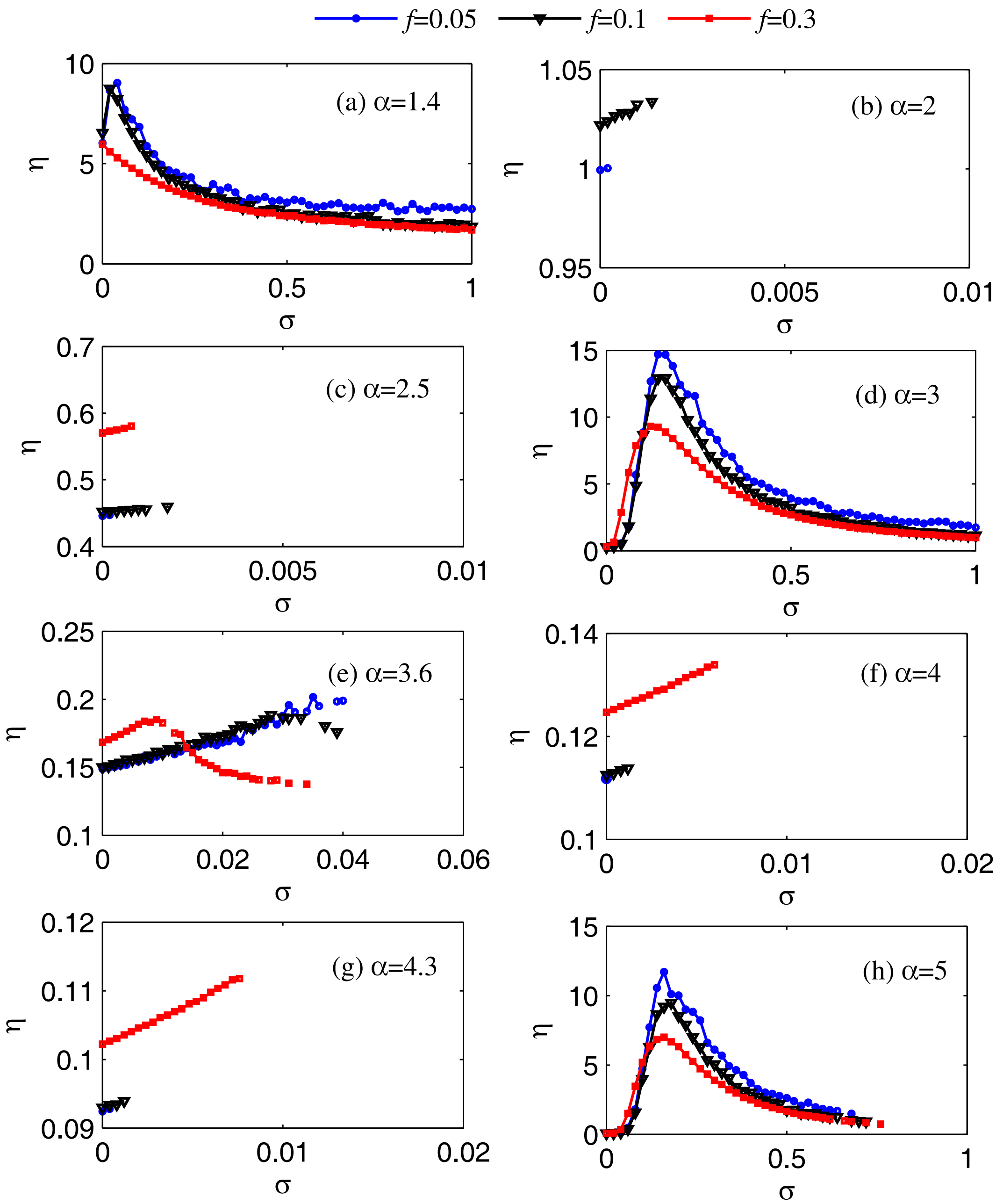


**Fig. 1.** The spectral amplification factor of the OGF system *versus* the noise intensity under different values of $\alpha$ and $f$. The simulation parameters are $a = 1$, $b = 1$ and $\omega = 0.1$.

In fig. 1(d), $\alpha = 3$, we have a typical bistable system. The curve of the spectrum amplification factor in fig. 1(d) is the most commonly SR curve. There are many references describing the SR phenomenon for this kind of system. In fig. 1(h), $\alpha = 5$, the system is a quintic oscillator with a bistable potential. The spectrum amplification factor *versus* the noise intensity presents the typical SR phenomenon in fig. 1(h). However, when we compare fig. 1(f) with fig. 1(d), we find that the system will be divergent in a much easier way for a stronger noise intensity. Further, the maximum of the spectrum amplification factor in fig. 1(h) is smaller than that in fig. 1(f). Moreover, in fig. 1, for smaller values of $f$, the spectrum amplification factor may have a larger maximum. As a result of fig. 1, the traditional bistable system is an excellent system to induce an SR phenomenon.

To indicate the SR phenomenon much more clearly, in fig. 2, we give the curves of the spectrum amplification factor *versus* the noise intensity under different values of $\alpha$ and $\omega$. The results in fig. 2 are similar to those in fig. 1. Moreover, for smaller values of $\omega$, the spectrum amplification factor has a much larger maximum.

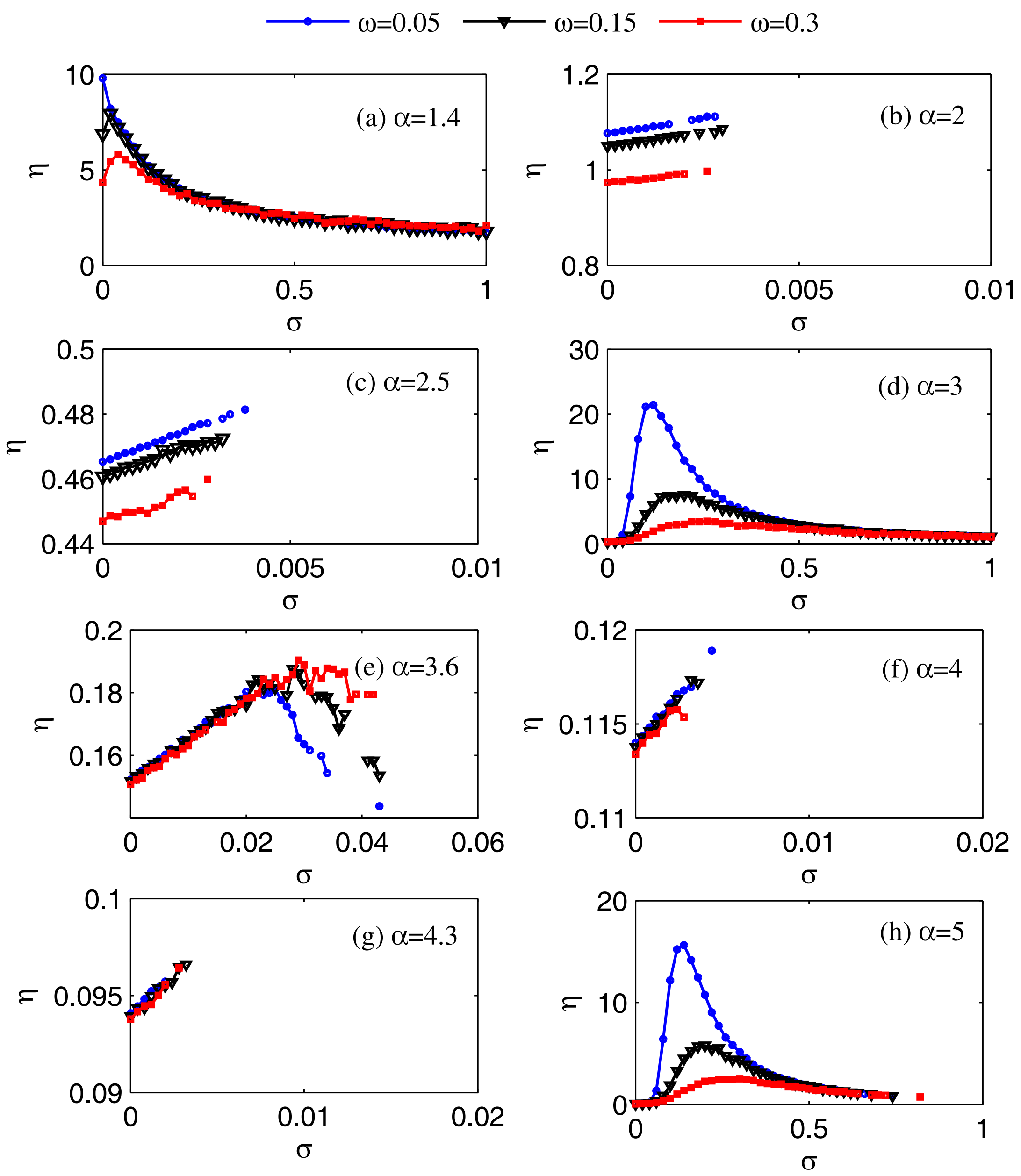


**Fig. 2.** The spectral amplification factor of the OGF system *versus* the noise intensity under different values of $\alpha$ and $\omega$. The simulation parameters are $a = 1$, $b = 1$ and $f = 0.15$.

### 2.2 Effect of the fractional exponent $\alpha$

In fig. 3, under different values of $\sigma$ and $f$, the curves of the spectrum amplification factor *versus* the fractional exponent $\alpha$ are plotted. A conspicuous property is the discontinuity on the curve. Specifically, the output of the system will be divergent in some discrete regions. In most of other regions, the spectrum amplification factor decreases with the increase of the noise intensity in each of these regions. When the fractional exponent $\alpha$ approaches to 1, the spectrum amplification has a large value. Moreover, for smaller values of $\sigma$, the spectrum amplification factor also has a larger value.

In fig. 4, under different values of $\sigma$ and $\omega$, the spectrum amplification factor $\eta$ *versus* the fractional exponent $\alpha$ are given. The results in fig. 4 are similar to that in fig. 3. Besides the discrete discontinuity regions on the curves, the spectrum amplification factor has larger values for the smaller values of $\sigma$ and $\omega$.

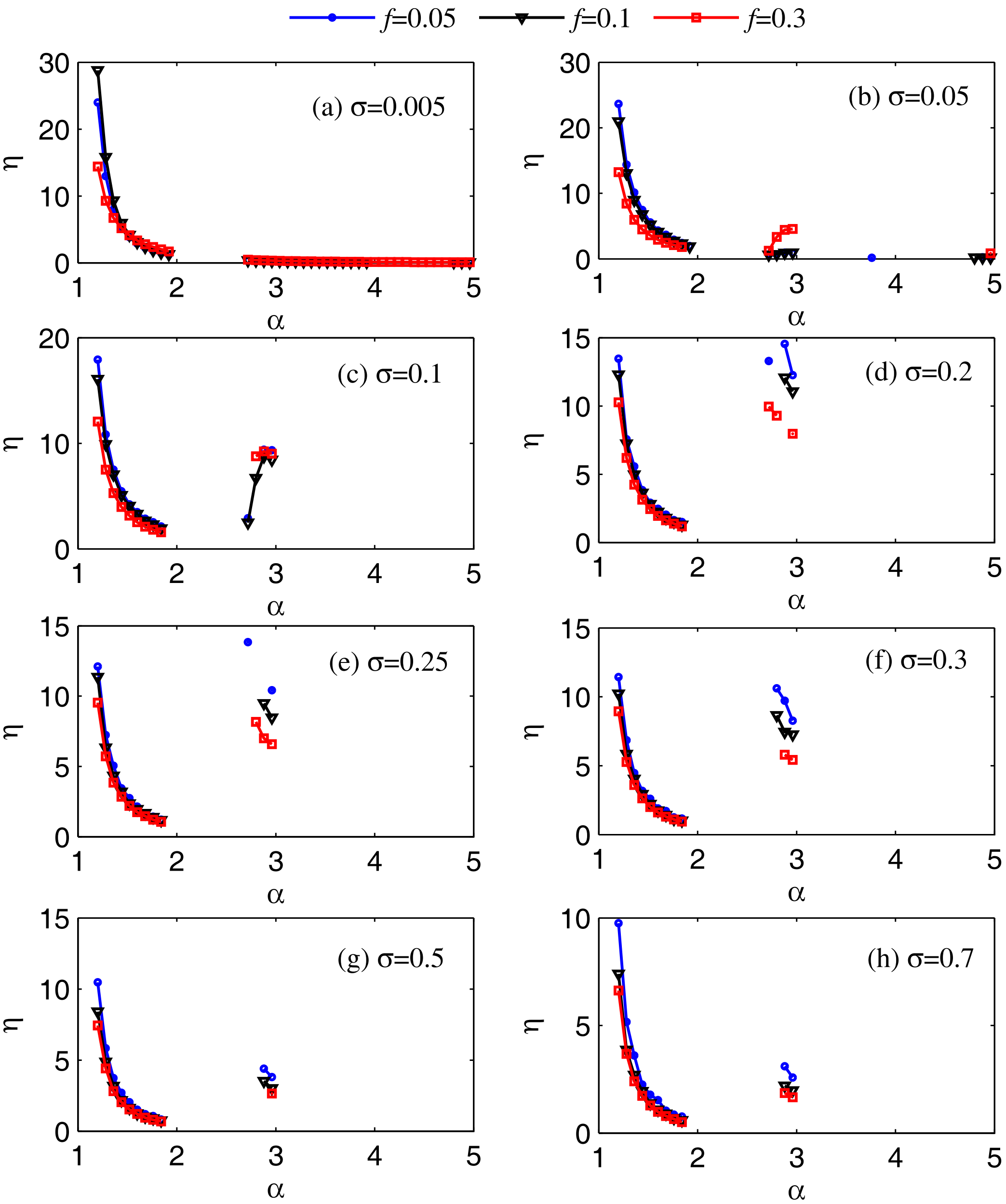


**Fig. 3.** The spectral amplification factor of the OGF system *versus* the power order under different values of $\sigma$ and $f$. The simulation parameters are $a = 1$, $b = 1$ and $\omega = 0.1$.

## 2.3 Effect of the nonlinearity

To investigate the effect of the nonlinearity, we plot in fig. 5 the spectral amplification factor *versus* the noise intensity under small values of the exponent $\alpha$. When $\alpha = 1$, the system is a stochastic linear system and there is no coupling element in the system. According to the stochastic dynamical theory, the spectrum amplification factor is not a nonlinear function of the noise intensity. Hence, there is no SR phenomenon when $\alpha = 1$. When the nonlinear term appears, *i.e.*, $\alpha > 1$, even though $\alpha = 1.05$, a slight SR phenomenon occurs. With the increase of $\alpha$, the SR phenomenon becomes more apparent. Hence, the nonlinearity is the necessary and key factor to induce SR. Further, we find that the peak value of the SR curve decreases with the increase of $\alpha$. Hence, in fig. 5, the nonlinearity is not the sufficient factor to amplify the weak low-frequency signal.

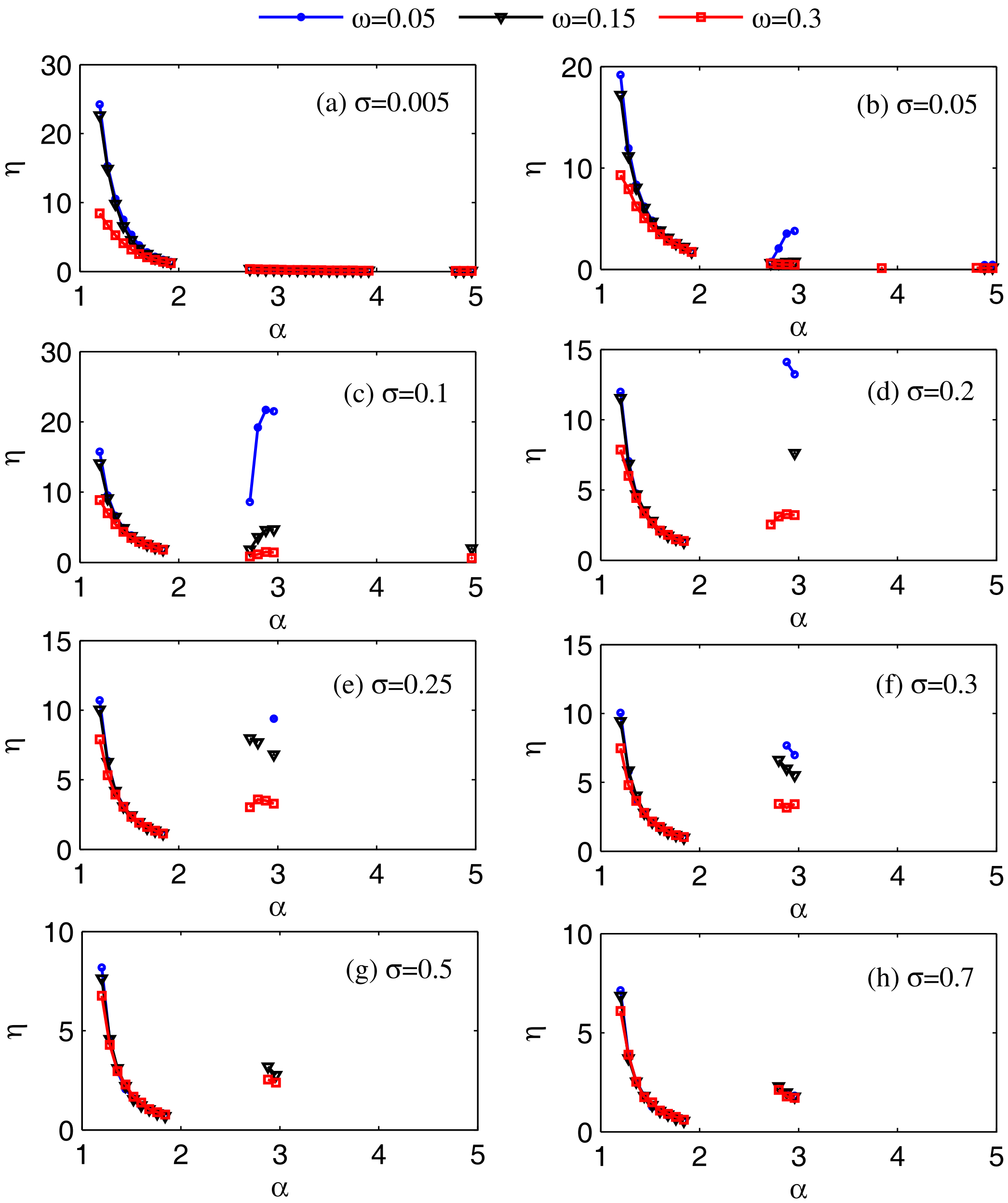


**Fig. 4.** The spectral amplification factor of the OGF system *versus* the power order under different values of $\sigma$ and $\omega$. The simulation parameters are $a = 1$, $b = 1$ and $f = 0.1$.

## 3 The SR phenomenon in the ODF system

Now, we study the SR phenomenon for the other kind of system, that is, the ODF system.

### 3.1 The noise induced SR

The spectral amplification factor *versus* the noise intensity in the ODF system is given in fig. 6. Very different from the $\eta$-$\sigma$ curves in fig. 1, the response is not divergent in fig. 6 for any value of $\alpha$. As a result, the ODF system is prior to the OGF system when it is considered as a SR system. In figs. 6(a) and (b), when $\alpha = 1.4$ and $\alpha = 2$, respectively, the SR disappears or turns unconspicuous for larger values of $f$, such as $f = 0.3$. In fig. 6, we also see that the spectral amplification factor may have a larger value for smaller values of $f$. In other words, the weaker signal is easy to be

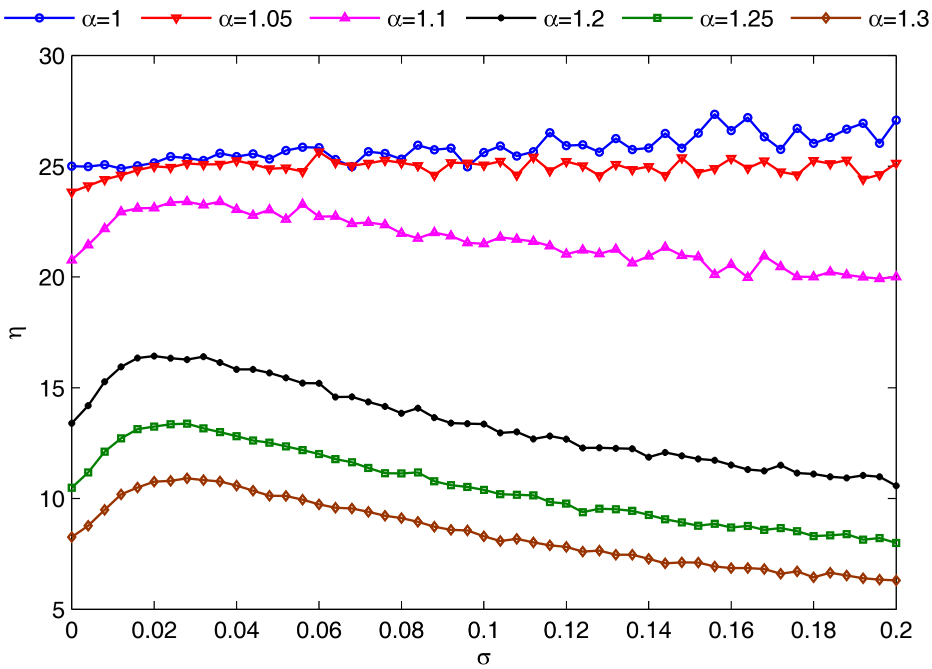


**Fig. 5.** The spectral amplification factor of the OGF system *versus* the noise intensity under different small values of $\alpha$. The simulation parameters are $a = 1$, $b = 1$, $f = 0.1$ and $\omega = 0.2$.

amplified. Moreover, when $\alpha$ turns from 1.4 to 5, the spectral amplification factor reduces gradually. We will research on the effect of the fractional exponent on the spectral amplification factor later.

The SR is displayed in fig. 7, and there is no divergent phenomenon for different values of $\alpha$. For lower signal frequencies, the response will achieve a higher resonance peak. It is a common property of the SR phenomenon. With the increase of $\alpha$, the spectral amplification factor will decrease. It is the same as what happens in fig. 6. If we compare fig. 7 with fig. 2, one can see that the ODF system has a better performance than the OGF system, once again.

### 3.2 Effect of the fractional exponent $\alpha$

In this subsection, we give the curves of $\eta$-$\alpha$ under some different simulations parameters, to make clear the effect of the fractional exponent $\alpha$ on the spectral amplification factor $\eta$ further.

The dependence of the spectral amplification factor $\eta$ on the fractional exponent $\alpha$ under different values of $\sigma$ and $f$ is shown in fig. 8. Differently from what appears in fig. 4, fig. 8 shows that the fractional exponent cannot induce divergence in the ODF system. It also reveals that the spectral amplification factor closely depends on the fractional exponent $\alpha$. Specifically, the spectral amplification factor will decrease with the increase of the fractional exponent. Under different values of $f$, this fact is always valid.

In fig. 9, under different values of $\sigma$ and $\omega$, the spectral amplification factor $\eta$ *versus* the fractional exponent $\alpha$ is given. In this figure, we find that the spectral amplification factor may achieve a larger value for weak signals with lower frequency. Moreover, for a weaker noise excitation, the spectral amplification factor may have a larger value.

As a conclusion of this subsection, we find that the ODF system has a better performance when SR occurs. This is mainly due to the fact that the OGF system induces easily a divergent response. It can also be explained from their potential functions. For the OGF system, the form of its potential function depends on the fractional exponent. For some cases, the potential function does not have a large enough attraction region. It leads to the response to be easy to be divergent for these cases. For the ODF system, the form of its potential is independent of the fractional exponent. Specifically, the potential function always has a bistable form. The attraction region for this case is large enough. The response is not easy to be divergent for this kind of potential. Consequently, it results that the ODF system has a better SR performance than the OGF system.

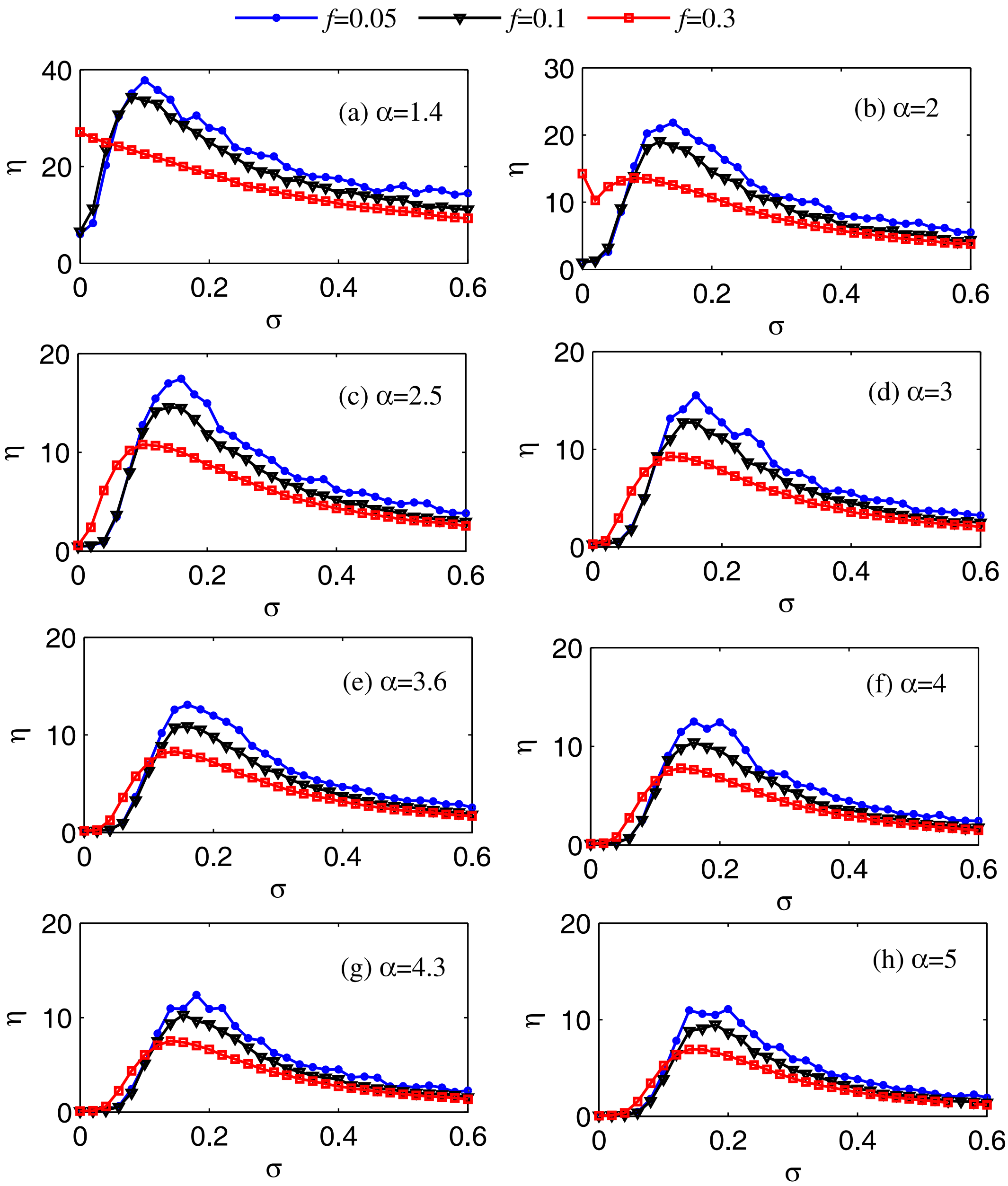


**Fig. 6.** The spectral amplification factor of the ODF system *versus* the noise intensity under different values of $\alpha$ and $f$. The simulation parameters are $a = 1$, $b = 1$ and $\omega = 0.1$.

### 3.3 Effect of the nonlinearity

Usually, the SR is investigated in the nonlinear system as we mentioned in the introduction. Hence, it is important to study the effect of the nonlinearity on the SR phenomenon. For the case $\alpha = 1$, the ODF system degenerates to a linear system. When $\alpha$ approaches 1, the system is approximate to a linear system. In fig. 10, when the fractional exponent $\alpha$ is very close to 1, such as $\alpha = 1, 1.05, 1.1$, there is no apparent SR phenomenon. With the increase of $\alpha$, such as $\alpha = 1.2, 1.25, 1.3$, the SR phenomenon is clearly observed. Hence, the nonlinearity is the necessary factor for the SR to occur. Another important fact is that the nonlinearity cannot amplify the weak signal. This is

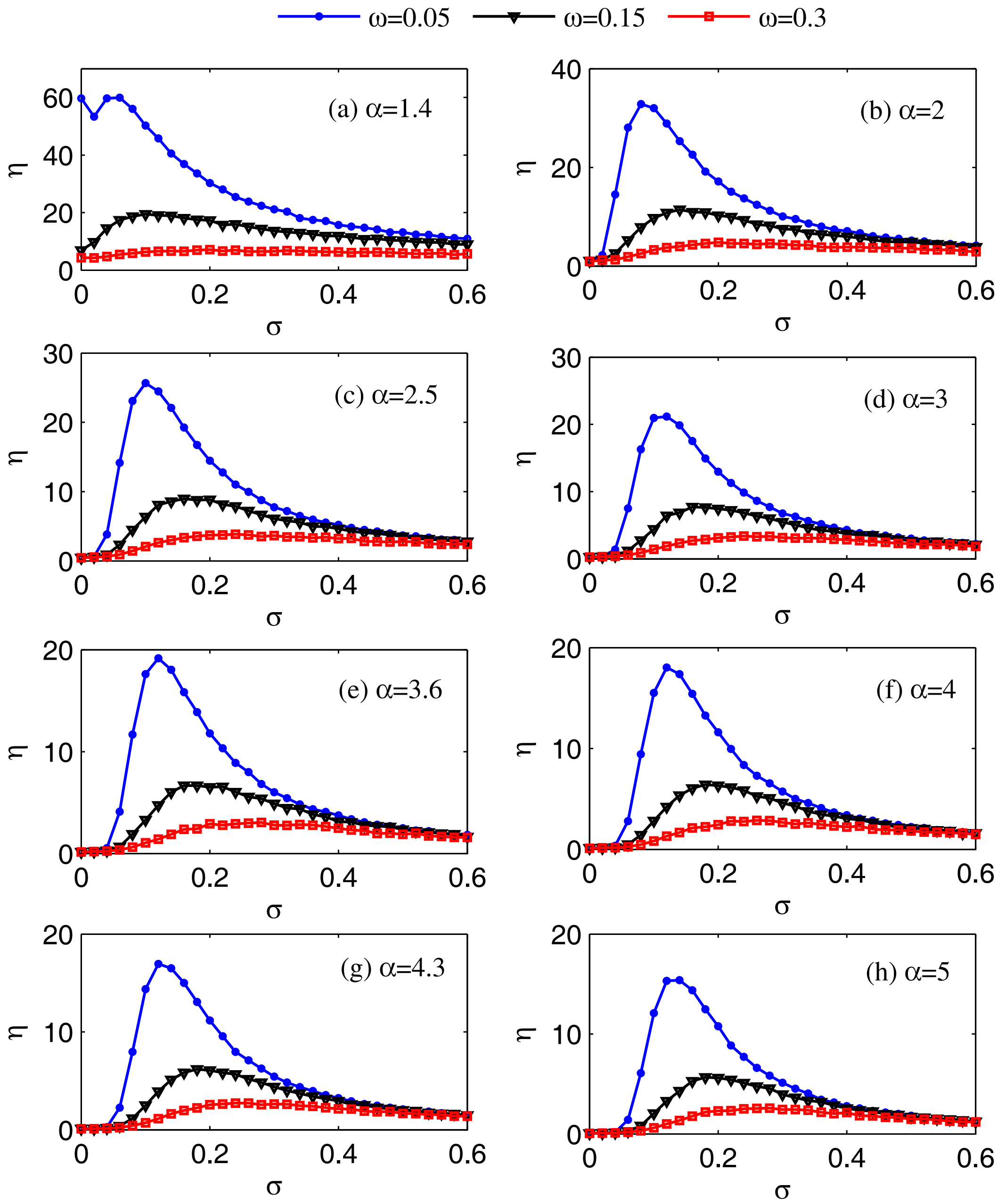


**Fig. 7.** The spectral amplification factor of the ODF system *versus* the noise intensity under different values of $\alpha$ and $\omega$. The simulation parameters are $a = 1$, $b = 1$ and $f = 0.15$.

because the spectral amplification factor will decrease with the increase of the fractional exponent $\alpha$. In other words, the spectral amplification factor obtained from the linear system is higher than that obtained from the nonlinear system in fig. 10. Hence, in fig. 10, we can say that the nonlinearity is the necessary ingredient to induce the SR phenomenon but not the sufficient condition to amplify the weak signal. The fact shown in fig. 10 is the same as that in fig. 5.

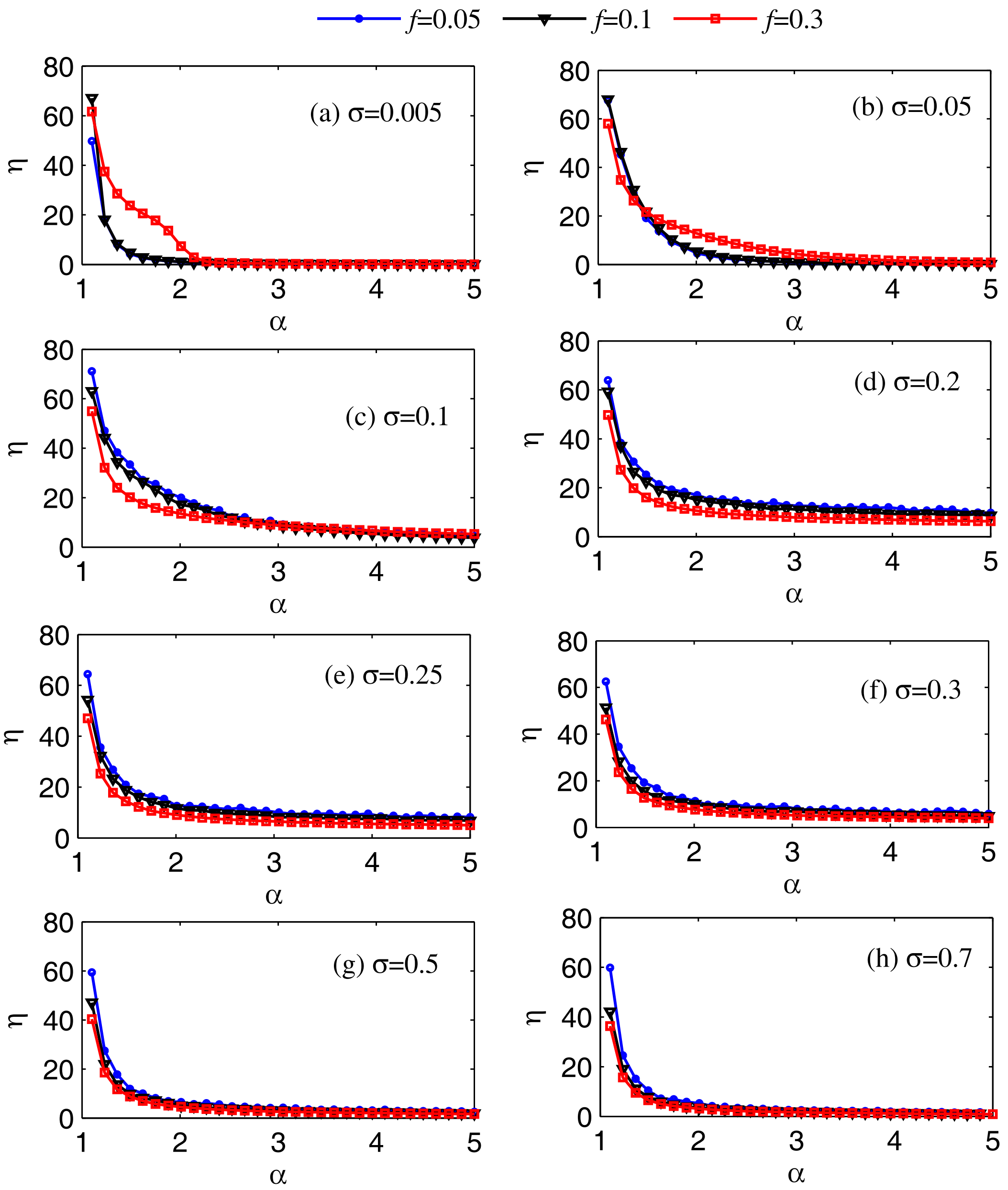


**Fig. 8.** The spectral amplification factor of the ODF system *versus* the power order under different values of $\sigma$ and $f$. The simulation parameters are $a = 1$, $b = 1$ and $\omega = 0.1$.

## 4 Conclusions

The classic SR phenomenon in the OGF and the ODF systems is investigated in detail. The OGF system is an overdamped system with a general fractional power nonlinearity. The ODF system is an overdamped system with a fractional deflection nonlinearity.

In the OGF system, the response of the system closely depends on the fractional exponent value. The response will be divergent for some regions of the fractional exponent values. In other words, the SR phenomenon may occur only in some discrete regions when the fractional exponent value is treated as a controllable variable. Moreover, the nonlinearity of the OGF system is the necessary factor to induce the SR phenomenon. However, it is not the sufficient factor to amplify the weak low-frequency signal. In general, the spectral amplification factor of the OGF system is lower than that of the corresponding linear system.

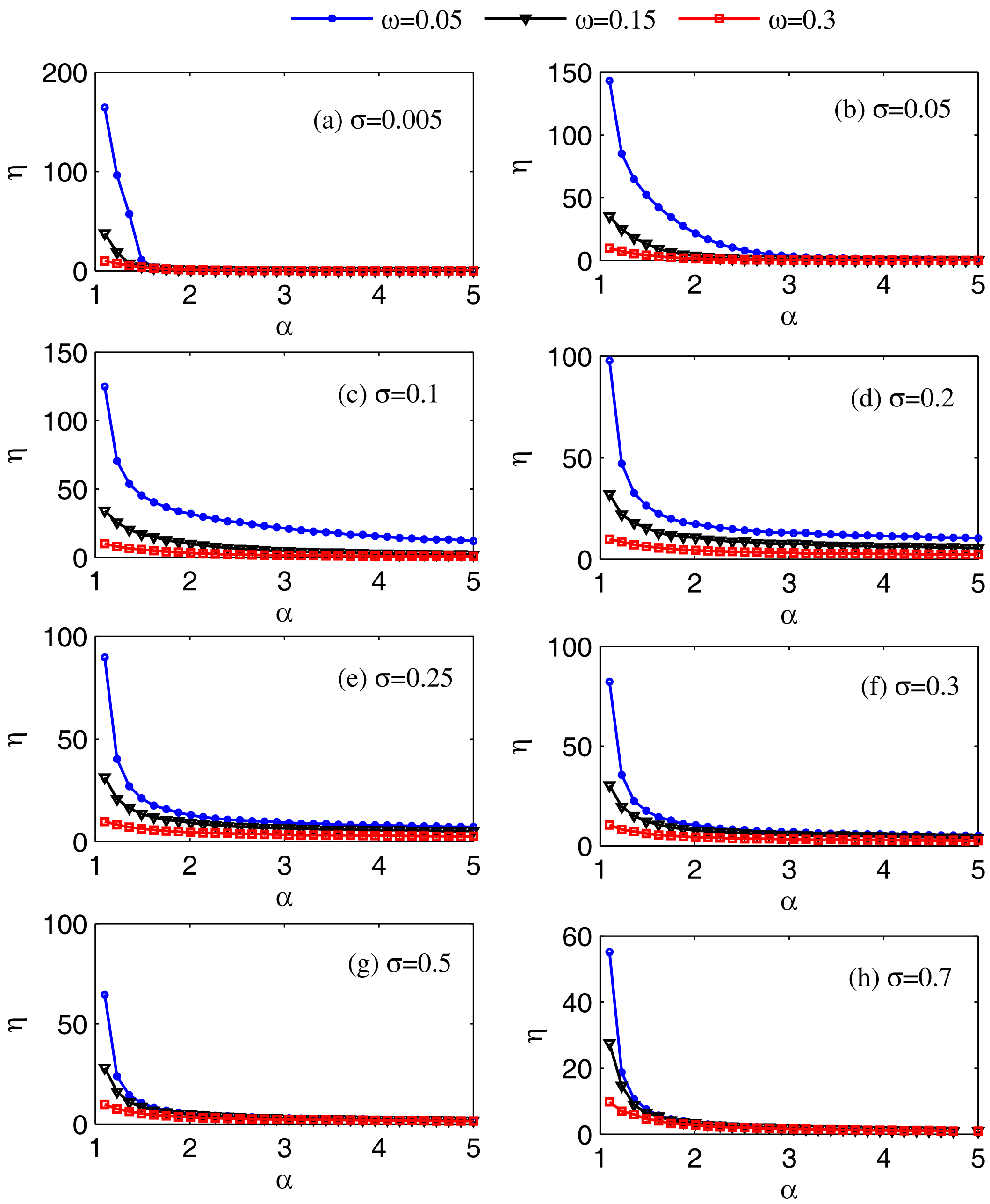


**Fig. 9.** The spectral amplification factor of the ODF system *versus* the power order under different values of $\sigma$ and $\omega$. The simulation parameters are $a = 1$, $b = 1$ and $f = 0.15$.

In the ODF system, the response of the system also closely depends on the fractional exponent. On the one hand, the response will not be divergent for any value of the fractional exponent. On the other hand, the spectral amplification factor *versus* the fractional exponent presents a decreasing function. Moreover, the nonlinearity is a necessary factor to induce SR, but it is not the sufficient factor to amplify the weak low-frequency signal. On this point, it is the same as that of the OGF system.

Through the analysis of the SR in nonlinear systems with different fractional power nonlinearities, we have found that the ODF system has a good performance. Further, when we chose an appropriate fractional exponent, we can have a better SR output than in the classic bistable system. The spectral amplification factor can be improved by it. We believe that our results can be useful in the field of signal processing. For example, when we use the ODF system to extract a weak signal in a noisy background, the extraction efficiency might be improved excellently.

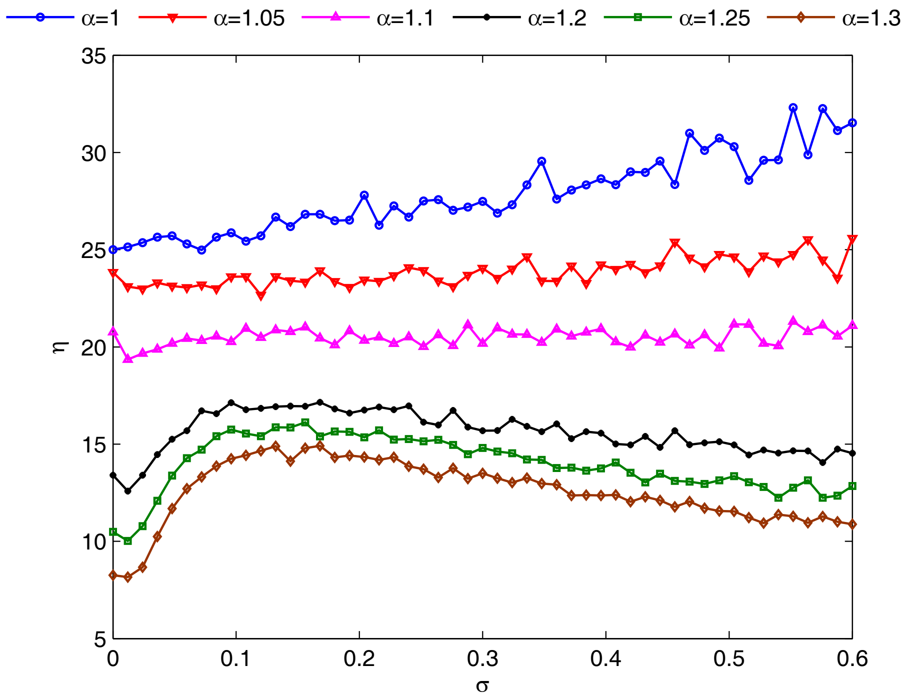


**Fig. 10.** The spectral amplification factor of the ODF system *versus* the noise intensity under small values of $\alpha$. The simulation parameters are $a = 1$, $b = 1$, $f = 0.1$ and $\omega = 0.2$.

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