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Infinite horseshoes and complex dynamics in physical systems

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ABSTRACT

Horseshoe maps have played an important role in the study of nonlinear dynamical systems. Here we study maps associated to two simple physical systems: a four-hill potential and an open billiard. They turn out to be very different from the standard horseshoe maps: one iteration stretches and folds a square in phase space an infinite number of times before placing it across itself. In this paper we explore in further depth these *infinite horseshoe* maps. We show that the infinite folding action requires that the maps are not defined in part of the rectangle. Our exploration also shows that infinite horseshoes provide valuable information on the complexity of the system. In particular, they provide a visual way to code the wide variety of complex orbits existing in the dynamical systems considered.

1. Introduction

Chaos theory traces back to the work of Poincaré [1], who wrote that if the stable and unstable manifolds of a saddle point of a map **f** cross, there is complicated behavior. Smale showed that in such cases there were always sets S which were topologically squares on which **f** (or $\mathbf{f}^n$) was a horseshoe map [2]. Since then, this kind of map has been frequently used to understand many different phenomena that are typical in nonlinear dynamical systems, from chaos itself to transient chaos [3], boundary crises [4] and fractal basin boundaries [5].

Horseshoe maps have been found in a wide variety of physical systems such as a bouncing ball in an oscillating table [6], a driven laser [7], a nonlinear oscillator [8], an atom–field interaction [9], a bistable optical system [10], a Josephson junction [11] and in fluid advection [12]. The presence of a horseshoe can be found explicitly or inferred by determining if the invariant manifolds intersect. Such intersections can also be deduced by means of the Melnikov method [13], which gives an analytical criterion for the existence of those intersections in a dynamical system.

However, we were surprised to find that a for couple of two-dimensional maps **f** arising naturally in two simple physical systems typically studied in the context of scattering problems, for a given square S in phase space $\mathbf{f}(S)$ crosses S an *infinite* number of times in a horseshoe-like manner. This kind of *infinite horseshoes* had been suggested first by Moser [14] and numerical examples have been found for the restricted three-body problem [15] and for a map arising in the study of solitary wave interaction [16]. We had previously found evidences of the existence of maps that produce an infinite number of crossings have for a chaotic scattering problems in the context of *partial control* of chaos [17,18], where finding a horseshoe-like map was a necessary condition to control the system [19,20] in order to avoid escapes of the trajectories with a control smaller than the noise (although new algorithms allow one to achieve this goal even without finding a horseshoe in phase space [21,22]). However in that paper we did not perform an exploration of this kind of dynamics. For this reason, the aim of this paper is to describe the conditions that allow the existence of infinite horseshoes, to provide two novel examples of infinite

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horseshoes that arise naturally in two well-known physical systems and to show how the existence of an infinite horseshoe in a physical system has interesting implications, implying the existence of an infinite variety of periodic trajectories.

This paper is organized as follows. In Section 2 we give some basic mathematical conditions that allow a continuous map to present an infinite number of foldings. Section 3 provides a definition of the two maps considered and in Section 4 we show that the geometrical action of these maps is quite complex, showing evidence of the infinite foldings. Symmetries of the system and possible itineraries corresponding to complex trajectories are shown in Section 5. The Section 6 is devoted to the conclusions of the paper.

2. Basic mathematical conditions for infinite foldings

The action of a standard horseshoe map on a square S is shown in Fig. 1. The map stretches and folds the square and places it across itself. Notice that in Fig. 1 if $\mathbf{f}(S)$ crosses twice $S, \mathbf{f}^2(S)$ would cross it 2^2 times and $\mathbf{f}^n(S)$ would cross it 2^n times. This kind of geometrical action has been proven to lead to the existence of an invariant set where the dynamics are chaotic. Although the conditions on how $\mathbf{f}(S)$ needs to cross S to have complex dynamics have been relaxed by Kennedy et al. [23], it is often assumed that the maps $\mathbf{f}$ of interest are continuous and defined in any square of the phase space S. This is a typical assumption for maps used in physics, because typically nearby trajectories should be mapped to near points and they should behave properly in the regions of interest. Mathematically, these two assumptions imply that $\mathbf{f}$ is uniformly continuous. Furthermore, uniform continuity implies that $\mathbf{f}(S)$ crosses S a *finite* number of times.

On the other hand, the existence of infinite horseshoes (maps stretching and folding an infinite number of times a square) has been reported in the three body problem and related with the first return map of a region close to a homoclinic orbit [14]. Here we want to enlighten some even more basic criterion that a map needs to fulfill to give rise to an infinite horseshoe. The basic idea is that if $\mathbf{f}$ is continuous, but is defined only on a part of S, then it is possible for $\mathbf{f}(S)$ to cross S infinitely many times.

In order to see this, a one-dimensional example that might help to understand how a map can present an infinite number of folds is the one-dimensional map $x_{n+1} = \sin(1/x_n)$. The graph of this map can be observed in Fig. 2. This map stretches and folds the interval $I = [-1/2, 1/2]$ an infinite number of times across itself. Note that the map is not defined at $x = 0$ and thus, although it is continuous elsewhere, it is not uniformly continuous. Recall that a function f is uniformly continuous in an interval I' if for any two sequences y_k and z_k in $I', |y_k - z_k| \to 0$ as $k \to \infty$ implies that $|f(y_k) - f(z_k)| \to 0$ as $k \to \infty$. Consider then two sequences of negative numbers $y_k \to 0$ and $z_k \to 0$ and satisfying $f(y_k) = 1/2$ and $f(z_k) = -1/2$ respectively. These sequences show that the map is not uniformly continuous, and thus infinite crossings become possible.

Thus, continuity and being only defined in certain regions of phase space can give rise to infinite crossings. In the next section, we show two examples in which maps fulfilling these conditions arise naturally, and for which infinite horseshoes can be easily found.

3. Maps for chaotic scattering problems

Now, we focus on the geometrical action of the maps arising in two simple scattering systems. The first map arises from a point moving in a four-hill potential given by $V(x,y) = x^2y^2exp(-(x^2+y^2))$, which has been studied in the investigation of chaotic scattering [24,25]. In this system, the key parameter is the energy E of the particle, which is always smaller than the height of the hills $V(\pm1, \pm1) = exp(-2)$. The second system is an open billiard that consists of two unit-radius hard disks that are separated by a distance d and with a reflecting hard wall under them at a distance $d/2$. Thus, in this system the key parameter is d. See Fig. 3.

For these systems, a map $\mathbf{f}$ can be obtained as follows. In the four-hill system, we observe a particle motion and if it crosses the x axis at least n times, x_n denotes the location of the crossing and θ_n is the angle from the vertical in $(-\pi/2, \pi/2)$. This angle is positive if the trajectory direction is in the right half plane. See Fig. 4(a). Furthermore, (x_n, θ_n) determines the trajectory and so (x_{n+1}, θ_{n+1}) determines it for the next crossing. Thus, we can write

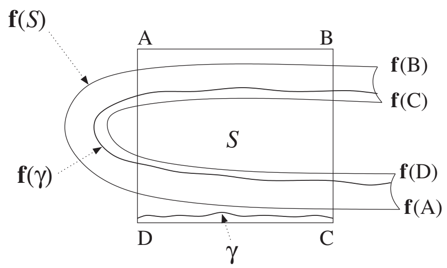

Fig. 1. A "standard" horseshoe map. This two-dimensional map **f** basically stretches and folds the square S before placing it across itself, in such a way that $\mathbf{f}(S) \cap S$ consists on two strips. This kind of geometrical action on the phase space is thought to be behind different complex behaviors in dynamical systems. We call **f** a "2-horseshoe" map because for each curve γ running from the left to the right of $S, \mathbf{f}(\gamma) \cap S$ similarly crosses S at least twice.

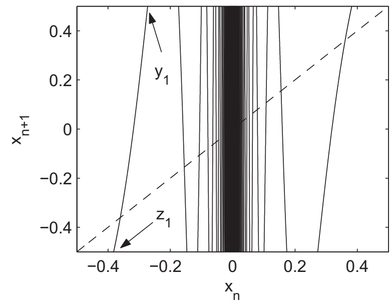

Fig. 2. The graph of the one-dimensional map $x_{n+1} = \sin(1/x_n)$ stretches and folds $I = [-1/2, 1/2]$ an infinite number of times, i.e., $f(I)$ crosses I an infinite number of times. This is related to the fact that the map is not uniformly continuous inside the rectangle.

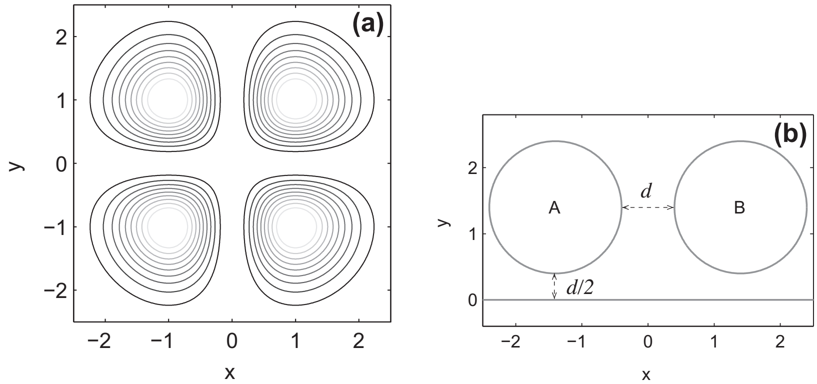

Fig. 3. The two systems considered here (a) The contour-plot of a four-hill potential $V(x, y) = x^2y^2exp(-(x^2 + y^2))$ for values of the energy E smaller than the "height" of the hills $E < exp(-2)$ and (b) an open billiard consisting on two unit-radius hard disks separated by a distance d and a reflecting hard wall placed at a $d/2$ distance units under them.

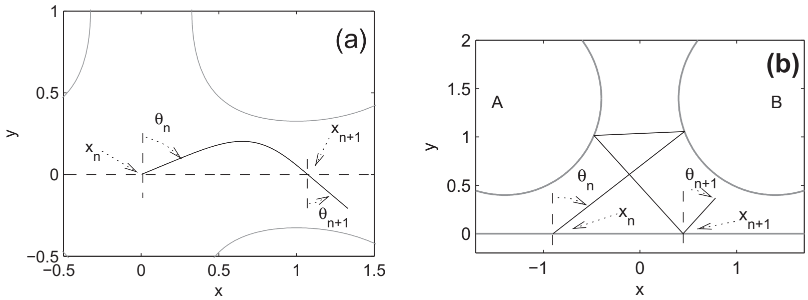

Fig. 4. For each of the systems considered, we can define a one-to-one map **f** as follows. (a) In the four-hill case, we consider the map that relates the position x_n and the angle from the vertical θ_n of a crossing of a given trajectory with the $y = 0$ line with the position x_{n+1} and angle θ_{n+1} of the next one (if any). Note that $\theta \in (-\pi/2, \pi/2)$. (b) In the two-disk system the map takes as arguments the position x_n and the angle θ_n of a reflection against reflecting wall, giving the position x_{n+1} and angle of reflection θ_{n+1} of the next one (if any). These maps are clearly not defined everywhere in the region defined by $\mathbb{R} \times (-\pi/2, \pi/2)$.

$$(x_{n+1}, \theta_{n+1}) = \mathbf{f}(x_n, \theta_n). \quad (1)$$

If there is no $n + 1$st crossing for this trajectory, then **f** is not defined in (x_n, θ_n).

Analogously, for a trajectory on the two-disk system, for the nth collision we write the position x_n and the angle of reflection from the vertical $\theta_n \in (-\pi/2, \pi/2)$, as shown in Fig. 4(b). These two numbers determine the trajectory and thus they determine the position and the angle of the $n + 1$st bounce (x_{n+1}, θ_{n+1}), if any. Thus, we can write an equation of the type of Eq. (1) that relates two consecutive collisions. Here again, if there is not a $n + 1$st collision, then the map will not be defined at (x_n, θ_n).

Thus, we have found two maps that can be computed numerically for two simple physical examples. Similar maps could be defined for a wide variety of physical systems. In the next section, we study the geometrical action of these maps.

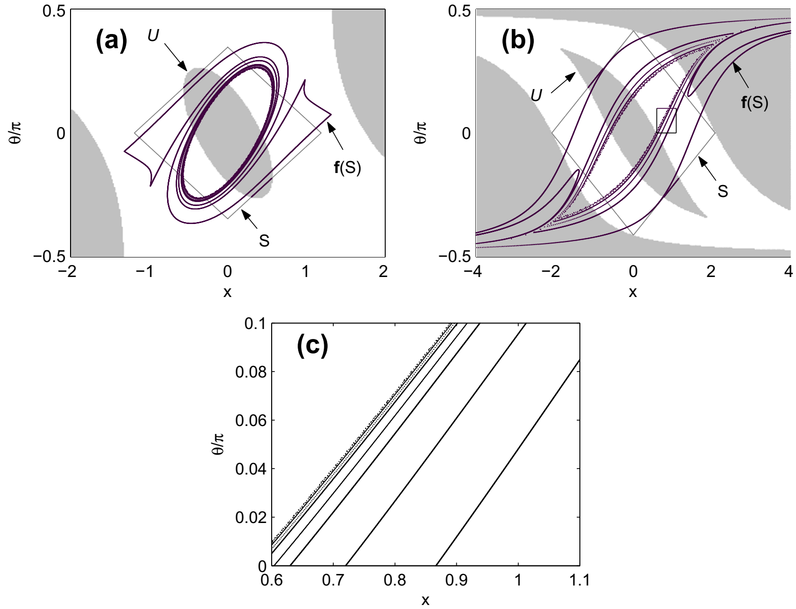

Fig. 5. For a parallelogram S (grey) in phase space we plot $\mathbf{f}(S)$ (black). The points with no image under $\mathbf{f}$ are marked in grey except those lying in the bounds of S, that are marked in black. The central region where $\mathbf{f}$ is not defined is called U. This is shown in (a) for the four-hill system with $E = 0.26exp(-2)$ and in (b) for the two-disk system with $d = 0.8$, (c) shows an enlargement of the square in (b) revealing an infinite number of lines that accumulate close to the reflection of U about $\theta = 0$. The structure of $\mathbf{f}(S)$ shows that the square S is stretched and then folded an *infinite* number of times. We call these maps *infinite horseshoes*.

4. Geometrical action of the maps

Thus, we have two maps whose phase space is $\mathbb{R} \times (-\pi/2, \pi/2)$. In order to visualize their geometrical action, we take a parallelogram S in $\mathbb{R} \times (-\pi/2, \pi/2)$, and we obtain $\mathbf{f}(S)$ by integrating the equations of motion of each system for points lying in its boundary. We also color grey the points where $\mathbf{f}$ is not defined.

The results of our calculations are shown in Fig. 5, for the four-hill case with $E = 0.26exp(-2)$, in Fig. 5(a) and for the two-disk system with $d = 0.8$ in Fig. 5(b). First, we can notice that, as expected, in these two figures there are regions in phase space where the maps are not defined. In particular, for both systems there is a set of points U where $\mathbf{f}$ is not defined that intersects S. Notice that the set U contains the point (0,0) for which the map $\mathbf{f}$ is not defined.

The most interesting part of the geometrical action of the maps $\mathbf{f}$ is that it apparently stretches and folds S an *infinite* number of times across itself. This becomes evident from the zoom of Fig. 5(b) that can be seen in Fig. 5(c), where an accumulation of strips can be clearly observed. Furthermore, the mapping is horseshoe-like: for any curve γ going from the bottom left side to the upper right side of S, $\mathbf{f}(\gamma) \cap S$ contains an infinite number of curves going from the bottom left side of S to the upper right side of S (even though $\mathbf{f}$ is not defined in $\gamma \cap U$). For the standard horseshoe map, $\mathbf{f}(\gamma) \cap S$ would consist of two curves. Thus, these maps found in these two simple physical systems can be considered *infinite horseshoes*.

Again, the fact that $\mathbf{f}(S)$ crosses S an infinite number of times does not contradict the finite crossings rule given at the beginning of this paper. It is both physically and mathematically intuitive that given a point (x, θ) in S that has an image under $\mathbf{f}$, a point sufficiently close in the phase space (and thus a sufficiently close trajectory in the physical system) must have its image under $\mathbf{f}$ close to $\mathbf{f}(x, \theta)$. Thus, continuity must be preserved where $\mathbf{f}$ is defined. However, the stronger condition of uniform continuity does not need to be fulfilled, because we have seen that there are regions of the phase space, such as U, where $\mathbf{f}$ is not defined. Therefore, $\mathbf{f}$ is not necessarily uniformly continuous in S and infinite crossings become possible. In the next section we show that this kind of infinite horseshoes provides a straightforward picture of the wide variety of possible trajectories that the physical systems considered can display.

5. Symmetries and complex itineraries

The symmetries of the systems give us important information about $\mathbf{f}$. We define the symmetry transformations $\rho_x(x, \theta) = (-x, \theta)$ and $\rho_\theta(x, \theta) = (x, -\theta)$. First, notice that both systems are symmetric about the y axes, so if a trajectory is possible its reflection about the y axis is possible too. Mathematically, this implies that if we set $\mathbf{p} \equiv (x, \theta)$:

$$\mathbf{f}(\mathbf{p}) = -\mathbf{f}(-\mathbf{p}). \tag{2}$$

We have chosen S so that $S = \rho_x S = \rho_\theta S = \rho_x \rho_\theta S$, so Eq. (2) explains the symmetric appearance of $\mathbf{f}(S)$.

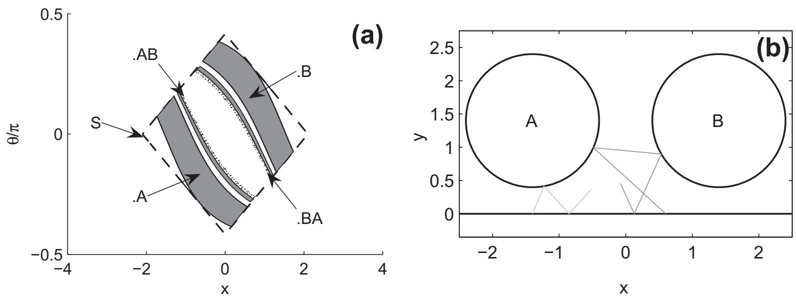

Fig. 6. Each one of the infinite strips appearing in $\mathbf{f}^{-1}(S) = \rho_\theta \mathbf{f}(S)$ (since $S = \rho_\theta S$) can be associated with a different type of trajectory. In (a) ".*A*" and ".*B*" denote regions whose trajectories collide with *A* and *B* respectively and then return to $y = 0$. The label of the regions gives then the sequence of bounces against *A* and *B* of their trajectories before coming back to $y = 0$. The ".*AB*" and ".*BA*" strips are also shown. In (b) we show an example of a ".*A*" trajectory (light grey) and other one of a ".*AB*" trajectory (grey). The existence of an infinite number of period *k* orbits, for each integer $k > 0$, can be inferred.

On the other hand, for both systems a time-reversed trajectory is a trajectory, which implies that:

$$\mathbf{f}^{-1}(\rho_\theta \mathbf{p}) = \rho_\theta \mathbf{f}(\mathbf{p}). \tag{3}$$

This equation implies that $\rho_\theta U$ has no image under $\mathbf{f}^{-1}$, and that is the reason why in Fig. 5 the infinite strips of $\mathbf{f}(S)$ accumulate around it.

Eq. (3), together with our numerical analysis, has important consequences from a dynamical point of view. A standard horseshoe map as in Fig. 1 maps two vertical strips in *S* to two horizontal strips, and this implies the existence of two period 1 orbits. For the infinite horseshoes, Eq. (3) implies that $\mathbf{f}^{-1}(S) \cap S$ is $\rho_\theta \mathbf{f}(S) \cap S$, the reflection of $\mathbf{f}(S) \cap S$ about $\theta = 0$, so we have an infinite number of pairs of strips, each of which is mapped to a strip (identical but transversal to it) in a horseshoe-like way. Thus, the infinite horseshoe map has an *infinite* number of period 1 orbits (analogously to the one-dimensional map of Fig. 2). The same applies to orbits of period *k*, whose number is finite for a standard horseshoe map and infinite for the infinite horseshoe map. Thus, these infinite horseshoes reveal that the dynamics that can be observed in these two simple dynamical systems is way more complex than the dynamics of a simple horseshoe-like map.

This dynamical complexity can be well understood using the following example: Periodic orbits of these maps are associated with periodic behaviors of the systems considered. Consider for example the two-disk system. Numerical simulations show that trajectories associated with each strip of $\mathbf{f}^{-1}(S) \cap S$ share common dynamical features. If we label the leftmost disk *A* and the rightmost one *B*, as shown in Fig. 3(b), it turns out that each strip determines the sequence of bounces against *A* and *B* of a trajectory before coming back to the wall. We can thus label each of these strips according to the sequence of bounces associated with them. The strips .*A*, .*B* and .*AB* are shown in Fig. 6(a), and two trajectories corresponding to .*A* and .*AB* are shown in Fig. 6(b). In fact, any alternate sequence of *A* and *B*, no matter how long it is, has a corresponding strip. This implies that there is a wide variety of periodic and nonperiodic behaviors in these systems that cannot be captured by a regular *finite* horseshoe map, which correspond to the trajectories that can bounce between the disk *A* and the disk *B* following arbitrarily long sequences of bounces between the disks before coming back to the plane for which the map is defined.

The example above also give us a hint of the origin of the absence of uniform continuity. In our infinite horseshoes, the arbitrarily thin strips can be labeled with strings of arbitrary concatenations of *AB*s (or *BA*s), which correspond to trajectories that do bounce an arbitrarily high number of times with both disks before coming back to the surface of intersection. Hence, it is not difficult to imagine a sequence of arbitrarily close trajectories that get closer and closer to those strips, which in turn lead to higher number of bounces with the two disks which eventually make that the small differences are amplified by the bounces. These trajectories provide the example of points arbitrarily close that can be mapped quite far, so the condition for uniform continuity is not met.

6. Conclusions and discussion

In conclusion, we have studied the maps associated with two simple physical systems. Simulations show that these simple maps stretch and fold regions in the phase space an infinite number of times, so they can be considered *infinite horseshoes*. Using a nonphysical example, we have illustrated that this infinite number of crossings is related with the fact that these maps are not defined everywhere in phase space, although they are continuous where they are defined. We have also shown that the presence of these horseshoes implies a wider number of dynamical behaviors than for the regular horseshoe maps. Thus we think that the graphical representation of the dynamics that these infinite horseshoes offer allows one to grasp the complexity of the dynamics of the systems considered as compared to other two-dimensional maps. On the other hand, proving mathematically the existence of an infinite horseshoe in a system like the ones considered here, is enough to prove that trajectories can be partially controlled, as long as it is known that the existence of a horseshoe is a sufficient condition for the existence of a partial control strategy [17,19] (although in practical terms, it is enough to use the existing algorithms in the system of interest to check if it is possible to partially control the trajectories [21,22]). We believe that infinite horseshoes can appear in a wide variety of contexts of interest in physics and that their study can shed light on interesting dynamical features of the system, in particular for the characterization of chaotic scattering problems [25].

Acknowledgements

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