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How to minimize the control frequency to sustain transient chaos using partial control

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ABSTRACT

In any control problem it is desirable to apply the control as infrequently as possible. In this paper we address the problem of how to minimize the frequency of control in presence of external perturbations, that we call disturbances, when the goal is to sustain transient chaos. We show here that the partial control method, that allows to find the minimum control required to sustain transient chaos in presence of disturbances, is the key to find such minimum control frequency. We prove first for the paradigmatic tent map of slope greater than 2 that for a constant value of the disturbances, the control required to sustain transient chaos decreases when the control is applied every k iterates of the map. We show that the combination of this property with the fact that the disturbances grow with k implies that there is a minimum control frequency and we provide a procedure to compute it. Finally we give evidence of the generality of this result showing that the same features are reproduced when considering the Hénon map.

1. Introduction

The study of transient chaos [1] is a major area of research in Nonlinear Dynamics. This is not surprising, provided that for any system with permanent chaotic behavior, it is possible to make this chaotic behavior transient by varying some of the system's parameters, for example through a boundary crisis [2]. The phenomenon of transient chaos is related with the existence of a zero-measure set in phase space, the chaotic saddle, inside which the dynamics is chaotic, but (contrarily to what happens with chaotic attractors) from which trajectories diverge. Thus, transient chaos can be formally related with a phenomenon of escaping dynamics: there exists a region Q enclosing the chaotic saddle from which nearly all trajectories *escape*.

Due to the widespread nature of this kind of behavior, different methods to *control* transient chaos by keeping trajectories inside Q have been proposed [3–6]. These methods aim to sustain the transient chaotic behavior, provided that this type of dynamics are desirable in different contexts such as species preservation [7,8] (where regular behavior is related to extinction) and even in engineering [6,9] (where regular behavior implies the misbehavior of an electric component or of an engine).

An important issue for any kind of control problem is the effect of *disturbances* such as noise, that typically make the control task more difficult. We have proposed recently a method that addresses the problem of disturbances when controlling transient chaos, referred to as *partial control of chaotic systems* [10–14]. Assuming that the effect of the external perturbations is bounded by certain constant ξ_0, that we refer to as *the disturbances*, this method allows one to keep trajectories inside the region of interest Q with a *control* that is always smaller than the disturbances. The method is called *partial control* because it

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does not tell how trajectories will behave inside Q, but it can guarantee that they will not escape from Q, so the transient chaotic behavior is sustained. This method was first applied to one-dimensional maps [15], and later generalized to maps with a horseshoe [16] in phase space [10,11,14]. Recently an algorithm to find *safe sets*, the key ingredient of the partial control method, has been found, and it allows to apply this method to any system with transient chaotic behavior [13].

Together with the amplitude of the control that needs to be applied, another important issue in any control problem is the *control frequency*, understood as the inverse of the maximum time between two consecutive applications of control (keeping the system controlled). For example, when trying to control the trajectory of a spacecraft, i.e., trying to make it reach certain target, the amplitude of control is determined by the maximum change in the velocity that we can obtain using the engines. The control frequency will determine how long we will be able to keep the spacecraft controlled, provided that we have a limited amount of fuel. Thus, in general, it is desirable to have the minimum possible control frequency. In absence of disturbances, the required frequency depends basically on how precise our knowledge of the state of the system is, and in principle it can be made as low as desired. However, the presence of disturbances makes things more difficult as long as increasing the time between two consecutive applications of control implies an increase of the degree of uncertainty on the future state of the system, mostly because the effect of the disturbances will grow in time.

We address here this problem in the context of control of transient chaos. Using ideas of the partial control method, we show that if there is a maximum value of the control that we can apply, say u_{max}, there is a minimum control frequency to keep trajectories inside the region of interest Q, thus sustaining the chaotic behavior. The paper is organized as follows: In Section 2 we show that this problem can be understood in terms of partial control of the kth iterate of a map with escapes and we describe the model that we use to derive our main results: the tent map. In Section 3 we show analytically that this model presents the interesting property that the control/disturbances ratio required to keep trajectories partially controlled decreases with k. Section 4 contains our key result. There we first characterize analytically the growth of the disturbances with k in our model. After this, we show that the combination of our results (the decrease of the control/disturbances with k and the growth of the disturbances with k) implies that there is a minimum control frequency, showing how to compute it. As expected, this minimum control frequency depends on the maximum control that can be applied on the system, u_{max}. Evidences of the generality of our results are provided in Section 5, where we show that similar features can also be reproduced with the paradigmatic Hénon map. In Section 6 we draw the main conclusions of our work.

2. Applying partial control every k iterates

2.1. Problem statement

Consider that we are trying to control a dynamical system with transient chaos, i.e., to prevent the escapes from a region Q where there is transient complex dynamics. The system might be affected by disturbances, and have equations of motion of the form $\dot{p} = f(p) + \xi(t)$, where $\xi(t)$ is some type of stochastic process of intensity σ. If the dynamics of a time-τ map or the Poincaré map in absence of disturbances are given by $p_{\tau+1} = f(p_\tau)$, in presence of disturbances it will be perturbed by a (random) amount $\xi(p_\tau, \sigma)$, so $p_{\tau+1} = f(p_\tau) + \xi_\tau(p_\tau, \sigma, \tau) \equiv f(p_\tau) + \xi_\tau$. In most situations, for moderate σ values, such ξ_τ will be bounded by a constant ξ_0.

We are interested in applying infrequent control perturbations to the system or, using the terminology given above, to apply perturbations every k iterates of the map with k as large as possible. If we consider the time-$k \cdot \tau$ map or the kth iterate of the associate Poincaré map, then the perturbed dynamics will be given by $p_{\tau+k} = f^k(p_\tau) + \xi_{k\tau}(p_\tau, \xi, k) \equiv f^k(p_n) + \xi_{k\tau}$ where $\xi_{k\tau}$ will now be bounded by certain $\xi_0(k)$, so $|\xi_{k\tau}| \leqslant \xi_0(k)$. We are not interested in the precise form of $\xi_{k\tau}$, we just need to know that the higher k is, the bigger $\xi_0(k)$ will be. By redefining the time index τ by an index n such that $n = k \cdot \tau$, we have that

$$p_{n+1} = f^k(p_n) + \xi_n, \tag{1}$$

where the new index n is the index that accounts for the dynamics of every k iterates of the map. Thus, applying control every k iterations can be represented mathematically as:

$$\begin{cases} q_{n+1} = f^k(p_n) + \xi_n & \text{where } |\xi_n| \leqslant \xi_0(k) \\ p_{n+1} = q_{n+1} + u_n & \text{where } |u_n| \leqslant u_0(k). \end{cases} \tag{2}$$

This is exactly the mathematical setting required to apply our partial control method [10–14]. Note that if f is a map with a chaotic saddle in a region Q from which nearly all trajectories escape, the same applies for f^k. The control that we apply to put the trajectory again on a safe set is u_n, that is applied every k iterates and that we assume that will depend on k. We also consider that the control is bounded by certain constant $u_0(k)$. We say that a set $S \subset Q$ is safe, if for each $p \in S$, the distance of $f^k(p) + \xi$ from S is at most u_0.

As we said, if f was the map associated to a flow with disturbances as sketched in the introduction, or a map where disturbances act every iteration, the effect of the disturbances will grow with k. This dependence on k is represented by $\xi_0(k)$, so $|\xi_n| \leqslant \xi_0(k)$. Note that $\xi_0(1) \equiv \xi_0$. It is important to emphasize that this approximation is valid under mild assumptions for the kth iterate of a map in which noise is applied each iteration, or for the kth iterate of a Poincaré map or a time$-\tau$ map of a flow. However, for certain Poincaré maps in presence of noise it might not be valid. For example, if we consider a

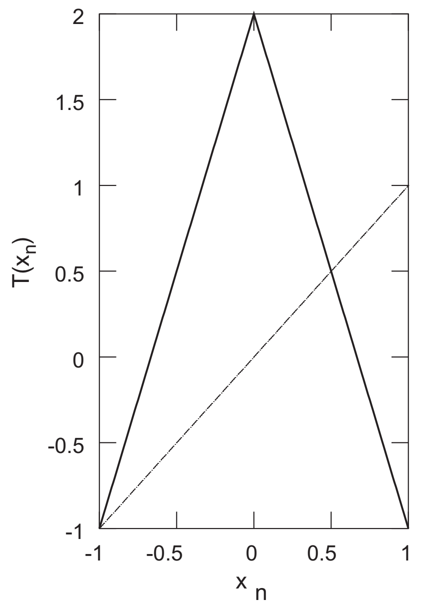

Fig. 1. The tent map with escapes $x_{n+1} = T(x_n)$ where $T(x) = \lambda(1 - |x|) - 1$ and $\lambda = 3$. There is a chaotic saddle in the [-1,1] interval, from which nearly all trajectories (except a zero-measure set) escape under iterations.

Poincaré map defined for certain chaotic scattering problems, we might have that for strong noise certain trajectories do not intersect k times with the Poincaré section. This would make impossible to define $\xi_0(k)$, so the discussion below does not apply for these situations. We believe though that a control method to minimize the control frequency in these situations could be devised using our ideas, but this is far from the scope of the present work.

In order to understand the effect of applying control every k iterations, first we are going to consider the effect of varying k while keeping $\xi_0(k)$ constant in a simple example of a dynamical system with escapes and transient chaos: the tent map. The results obtained will be a key ingredient in order to minimize the control frequency in the situations considered.

2.2. Our model: the tent map

We consider here in detail the effect of applying the control scheme provided in Eq. (2) to the kth iterate of the tent map of slope $\lambda > 2, T(x) = \lambda(1 - |x|) - 1$. From now on, we keep $\lambda = 3$ fixed: the resulting map is shown in Fig. 1. Recall that points that do not diverge to infinity in the limit $k \to \infty$ under T^k form the "middle-third" Cantor set built using the $[-1, 1]$ interval as initial segment: these points are the chaotic saddle of this system.

The map T is a good example of a map for which partial control can be applied. This map presents escapes from a region Q, the $[-1, 1]$ interval, that encloses a chaotic saddle with a well-characterized complex dynamics.

We want to study the role of applying the control every k iterates. As we said above, this is equivalent to the following control problem:

$$\begin{cases} q_{n+1} = T^k(x_n) + \xi_n \\ x_{n+1} = q_{n+1} + u_n, \end{cases} \tag{3}$$

with $|\xi_n| \leqslant \xi_0(k)$ and $|u_n| \leqslant u_0(k)$. In what follows, we are going to consider the role of k in this control problem while keeping $\xi_0(k)$ constant. The results will then be used to find the minimum control frequency needed in this type of problems.

3. Partial control of the *k* iterate of the tent map

3.1. The $k = 1$ case

The $k = 1$ case of Eq. (3) was thoroughly studied in Ref. [15]. There is shown that the sets $S^m \equiv T^{-m}(0)$ work as suitable safe sets for different values of ξ_0. This means that trajectories of points $\{x_n\}_{n=1}^{\infty}$ can be kept close to this set with $u_0(1) < \xi_0(1)$. The points of the safe set S^m are of the form:

$$\pm\frac{2}{3}\pm\frac{2}{3^2}\pm\frac{2}{3^3}\pm\cdots\pm\frac{2}{3^m}. \tag{4}$$

The key property, that makes them safe sets, is that each point of S^{m-1} has one point of S^m placed $2/3^m$ away to its left, and a point of S^m placed $2/3^m$ to its right (those given for the last "±" sign).

In Ref. [15] is shown that for $\xi_0 = 4/3^m$ the safe set that minimizes the required u_0 is given by S^m. In order to see this, consider for example that $\xi_0 = 4/3$. As we said, this implies that the adequate safe set is S^1, which reads

$$S^1 = T^{-1}(0) = \left\{-\frac{2}{3}, \frac{2}{3}\right\}. \tag{5}$$

The points of S^1 are displayed in Fig. 2(a). Note that $T(S^1) = 0$, so the image of S^1 under T has one point of S^1 to its left, $-2/3$, and another to its right, $2/3$. This property of the image of the safe set being "surrounded" by the safe set itself is the good property that allows one to keep trajectories on them with a control smaller than the disturbances. The reason is the following: assume that the first point of the trajectory x_1 is on S^1. Then no matter what the disturbance ξ_1 is, a control u_1 smaller than $\xi_0(1)$ can put the trajectory back on S^1. In particular:

- If $0 \leqslant |\xi_1| \leqslant \frac{2}{3}$, a correction of amplitude $|u_1| \leqslant \frac{2}{3} - |\xi_1| \leqslant \frac{2}{3}$ can steer the trajectory to a point on S^1.
- If $\frac{2}{3} \leqslant |\xi_1| \leqslant \frac{4}{3}$, a correction of amplitude $|u_1| \leqslant \xi_0 - \frac{2}{3} \leqslant \frac{2}{3}$ can take the trajectory to a point on S^1.

These situations are illustrated in Fig. 2(a). After applying the accurate correction u_1, we can make x_2 lie in S^1 and this can be repeated forever. Thus, we can estimate from the above considerations the control/disturbances ratio required to keep trajectories in $[-1, 1]$ for $\lambda = 3$ and $k = 1$ iterates of the map,

$$\frac{u_0(1)}{\xi_0(1)}\Big|_{\xi_0(1)=4/3} = \frac{\max(|u_1|)}{\max(|\xi_1|)} = \frac{1}{2}. \tag{6}$$

We will show later that this is actually the minimum control/disturbances ratio for $\xi_0(1) = 4/3$. Due to the self-similarities of the sets S^m (S^m consists of two small-scale copies of S^{m-1}, etc.. . .) we can see that for $\xi_0(1) = 4/3^m$ trajectories can be kept inside Q with a control that is exactly half the value of disturbances, bounded by $u_0(1) = 2/3^m$. On the other hand, for large values of the disturbances, (larger than the typical size of the chaotic saddle) the control/disturbances ratio is also smaller than one, and tends asymptotically to one as we increase the value of the disturbances.

We want to point out that for values other than $\xi_0(1) = 4/3$, it is also possible to keep trajectories bounded with a control that is smaller than the disturbances. This typically requires though other safe sets different from S^1, that can be computed making use of our Sculpting Algorithm [13]. These sets turn out to be preimages of an interval around 0. Using this simple idea, we can see that the result using other disturbances values will be qualitatively similar to the one observed for $\xi_0(1) = 4/3$ and trajectories can be kept bounded with $u_0(1) < \xi_0(1)$.

3.2. The $k \geqslant 1$ case

We consider now how to apply the partial control method for $k \geqslant 1$ and for the value of $\xi_0(k) = 4/3$, the same that for $k = 1$. A good guess would be to use the set $S^k = T^{-k}(0)$. First, because by definition, we can see that the points of S^k are mapped under T^k to 0, a point that again has points of the set S^k to its left, and points of the set S^k to its right. Thus, as in the $k = 1$ case, the image under T^k of $S^k, T^k(S^k) = 0$, is "surrounded" by S^k, so S^k is an adequate safe set. The strategy with this

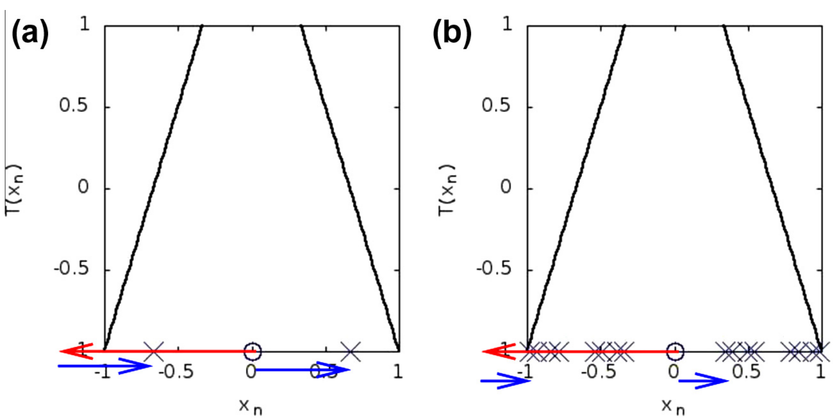

Fig. 2. Examples of situations that can arise by applying the partial control strategy for the slope-three tent map and for $\xi_0 = 4/3$. Points belonging to safe sets are marked with a '×', the zero is marked with a '∘'. (a) Control needed (blue arrows) to get the trajectories back on the safe set S^1 for $k = 1$, or $\xi_0 = 4/3$ in two extreme situations, with maximum disturbances (red arrow) and no disturbances. (b) The same but for the $k = 4$ case, using as a safe set S^4. Note that in both cases the control needed is smaller than the disturbance ξ_0, but that for $k = 4$ the control needed is smaller than for $k = 1$. (For interpretation of the references to colour in this figure caption, the reader is referred to the web version of this article.)

set would be the following: We choose x_1 in S^k, then $T^k(x_1) = 0$. No matter what the value of the disturbances ξ_1 is, we only have these possibilities:

- If $0 \leqslant |\xi_1| \leqslant \frac{2}{3} - \frac{2}{3^2} - \frac{2}{3^3} - \cdots - \frac{2}{3^k} = \frac{1}{3} + \frac{1}{3^k}$, a correction of amplitude $|u_1| \leqslant \frac{1}{3} + \frac{1}{3^k} - |\xi_1| \leqslant \frac{1}{3} + \frac{1}{3^k}$ can steer the trajectory to a point on S^k.
- If $\frac{1}{3} + \frac{1}{3^k} \leqslant |\xi_1| \leqslant \frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \cdots + \frac{2}{3^k} = 1 - \frac{1}{3^k}$, a correction of amplitude either $\frac{2}{3^k}$ or $\frac{4}{3^k}$ (the two possible values of half the distance between two consecutive negative or positive points of S^k) can take the trajectory to a point on S^k.
- If $1 - \frac{1}{3^k} \leqslant |\xi_1| \leqslant \frac{4}{3}$, a correction of amplitude $|u_1| \leqslant \xi_0 - (1 - \frac{1}{3^k}) \leqslant \frac{1}{3} + \frac{1}{3^k}$ can take the trajectory to a point on S^k.

Two of these possibilities are illustrated in Fig. 2(b) for $k = 4$. Thus, again, after applying the control u_1, smaller than $\xi_0(k)$, we can make the next point of the trajectory x_2 to lie on a point on the safe set S^k and this can be repeated forever.

Using the above considerations the control/disturbances ratio $u_0(k)/\xi_0(k)$ needed to keep the trajectories bounded for the k iterate of the tent map of slope $\lambda = 3$ and for $\xi_0(k) = 4/3$ is

$$\frac{u_0(k)}{\xi_0(k)}\Big|_{\xi_0(k)=4/3} = \frac{\max(|u_1|)}{\max(|\xi_1|)} = \frac{\frac{1}{3} + \frac{1}{3^k}}{\frac{4}{3}} = \frac{1}{4} + \frac{1}{4 \cdot 3^{k-1}}. \tag{7}$$

Clearly, its value decreases with k. In particular, we can see that as $k \to \infty$

$$\lim_{k\to\infty} \frac{u_0(k)}{\xi_0(k)}\Big|_{\xi_0(k)=4/3} = \frac{1}{4}. \tag{8}$$

Thus, we can see that by increasing k while keeping $\xi_0(k)$ constant, the control/disturbances ratio required to sustain transient chaos decreases, and for high k values trajectories can be partially controlled with a control $u_0(k)$ that is 25% of the disturbances $\xi_0(k)$. We have calculated numerically the minimum control/disturbances ratio needed for each value of k by using the Sculpting Algorithm [13] for the value $\xi_0(k) = 4/3$ and we have compared with the theoretical value given by Eq. (7). The results are shown in Fig. 3, showing a very good agreement with our theoretical result.

As for $k = 1$, we can discuss now briefly the effect of using other values of the disturbances. As in the $k = 1$ case, due to the self-similarities of the safe sets it can be seen that for values of $\xi_0(k) = 4/3^m$ the relation given by Eq. (7) holds. A similar behavior would be observed for other values of the disturbances, so we can conclude that the decrease of $u_0(k)/\xi_0(k)$ with k seen in Fig. 3 will be observed for any value of $\xi_0(k)$. In what follows we investigate further the generality of this result.

3.3. The effect of changing the slope

The results above hold for the tent map of slope $\lambda = 3$. It is not very difficult to see that similar results hold for values of λ above the critical value $\lambda = 2$, where the map possesses a boundary crisis. In particular, a decrease on the control/disturbances ratio needed to sustain transient chaos at $\xi_0(k)$ will also be observed. The way to see this is that, as for $\lambda = 3$, for a given $\lambda > 2$ and for values of the disturbances $\xi_0(k) = (2\lambda - 2)/\lambda$ the sets $S^k = T^{-k}(0)$ are safe sets. Furthermore, it can be proved that the asymptotic value of the ratio as k increases depends on the slope λ

$$\lim_{k\to\infty} \frac{u_0(k)}{\xi_0(k)}\Big|_{\xi_0(k)=2-2/\lambda} = \frac{\lambda - 2}{2\lambda - 2}. \tag{9}$$

This is confirmed again in Fig. 3, where the minimum control/disturbances ratio obtained theoretically using these safe sets for different λ values and those calculated numerically with the Sculpting Algorithm [13] are shown. Note that this implies

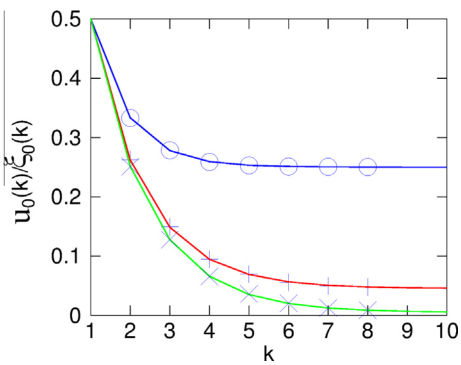

Fig. 3. Numerical estimation of the minimum control/disturbances ratio $u_0(k)/\xi_0(k)$ needed to keep trajectories bounded for the kth iteration of the tent map for values of the slope $\lambda = 3$ (○), $\lambda = 2.1$ (+) and $\lambda = 2.01$ (×). For each value of λ considered a constant disturbance $\xi_0(k) = 2 - 2/\lambda$ is used. Solid lines correspond to theoretical values. The asymptotic value obtained decreases as we get closer to the critical value $\lambda = 2$. Similar results are obtained for other values of the disturbances.

that as λ approaches 2, the ratio goes to zero. The main reason for this is the fact that as λ gets close to 2, the gap through which trajectories escape from the interval $[-1,1]$, that is the interval (s) that is (are) mapped out of it under T (T^k) becomes narrower and points in the safe sets are closer from each other as k increases. This result is also confirmed by the intuition that a slower escaping dynamics should imply that the control/disturbances ratio needed to avoid escapes is smaller.

For other values of the disturbances, we can see that due to the self-similarity properties of the safe sets the above results will hold for disturbances of the form $\xi_0(k) = (2\lambda - 2)/\lambda^m$ and that qualitatively similar results would hold for other values of $\xi_0(k)$ (although, as for $\lambda = 3$ they will require other safe sets). With this idea in mind, we can show that this property enables us to determine a minimum control frequency for our control problem.

4. A minimum control frequency

4.1. Increase of the disturbance with k

The above results characterize the behavior of the control/disturbances ratio $u_0(k)/\xi_0(k)$ assuming that our control problem is as described by Eq. (3) and taking $\xi_0(k)$ constant, i.e., $\xi_0(k)$ does not depend on k for different values of k. We have concluded that this ratio decreases with k to an asymptotic value that depends only on the system's parameters. However, as we noted in Section 2, when we are dealing with the kth iteration of a map in which every iteration we apply the noise, or when we are considering a map associated to a flow affected by the noise, $\xi_0(k)$ is not constant: in principle it will increase with k. In other words the choice of k, the number of iterates of our map (or the number of time-τ maps considered, or the number of intersections of the Poincaré section considered in our flow) will have an influence in the value of $\xi_0(k)$, something that we represented by the dependence on k of $\xi_0(k)$.

For a given dynamical system with a largest Lyapunov exponent L, we can expect that $\xi_0(k)$ will grow with k following the equation

$$\xi_0(k) \propto e^{kL}. \tag{10}$$

In fact, we have verified that this is exactly the case when considering the slope-λ tent map. For this map the largest Lyapunov exponent is $\log \lambda$ so the above equation becomes

$$\xi_0(k) \approx \xi_0 \lambda^{k-1}. \tag{11}$$

In Fig. 4 we show the numerical estimates of $\xi_0(k)$ for different ξ_0 values, computed as the maximum divergence of the uncontrolled trajectories from their deterministic path for every k. We see that the above expression provides a good estimate of $\xi_0(k)$ and confirms that it grows exponentially with k. In general, then, $\xi_0(k)$ will grow with k and this implies that the frequency of the control cannot be arbitrarily low, but it can be minimized. We have now all the ingredients to show how to achieve this goal.

4.2. Finding the minimum control frequency

Up to know, we have shown an interesting property for the tent map: that if we increase the value of k while keeping $\xi_0(k)$ constant, the control/disturbance ratio needed to keep the trajectories controlled decreases. If the disturbances were constant with k there would not be a minimum control frequency: in principle the most advantageous thing to do would be to apply the perturbations as unfrequently as possible, because this would minimize the required control. However,

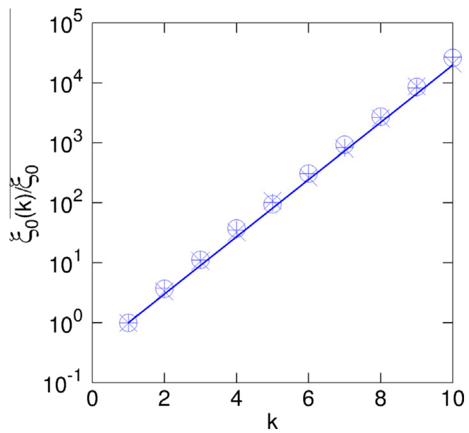

Fig. 4. Numerical estimation of the ratio $\xi_0(k)/\xi_0$ vs the number of iterations k. Notice that the $\xi_0(k)/\xi_0$ axis is in logarithmic scale. We have used the values $\xi_0 = 10^{-2}$ ('∘'), $\xi_0 = 10^{-1}$ ('+') and $\xi_0 = 1$ ('×') in the slope-three tent map. The solid line indicates the numerical estimation given by Eq. (11).

we have just shown that in general when we increase k the value of $\xi_0(k)$ will also increase, and this will make necessary a bigger control. From this trade-off emerges the main practical implication of our results, that can be summarized as follows: using the partial control strategy we can derive a minimum control frequency that is determined by u_{max}, the maximum control that we can apply to our system.

The way to determine this minimum control frequency is the following: we know that the control/disturbances ratio at fixed ξ_0 decreases with k. On the other hand, we know that the value of $\xi_0(k)$ grows with k. Thus, for every value of k we need an estimate of $\xi_0(k)$. With this value, we calculate the minimum control required to keep the trajectories bounded for this value of $\xi_0(k)$, i.e., the required $u_0(k)$ using the partial control strategy. Provided that $\xi_0(k)$ now grows with k, the value of $u_0(k)$, although always smaller than $\xi_0(k)$, will grow with k. We recall that the computation of $u_0(k)$ can be done automatically with our Sculpting Algorithm [13]. Then, the minimum control frequency will be determined by the bigger k such that

$$u_0(k) \leqslant u_{\max}, \tag{12}$$

where u_{max} is the maximum control that we can apply to our system. If we apply our controlling perturbation u every k iterates we can be sure that transient chaos can be sustained in our system. We want to emphasize that provided that the partial control method gives the minimum control/disturbances ratio $u_0(k)/\xi_0(k)$ needed to keep trajectories bounded (i.e., for every k we get the smaller $u_0(k)$ needed) we can be sure that the control frequency that we obtain following this procedure is the minimum possible one.

As an example of the above procedure, consider that we are dealing with the slope 3 tent map and that the maximum control that we can apply is $u_{max} = 0.5$. Consider as well that every iteration the map is affected by a disturbance bounded by $\xi_0 = 0.01$. Of course, it would not be difficult to keep trajectories bounded with frequent perturbations, i.e., applying a control every iteration to the system. However, we want to know what is the smallest frequency allowing us to keep trajectories controlled using always a control that is smaller than our bigger allowed control, u_{max}. To do this, we can use our estimate of the effect of disturbances after k iterations, $\xi_0(k)$, that we already have shown in Fig. 4. Then, with each value of $\xi_0(k)$ we have to compute the safe set that requires the minimum $u_0(k)$ using the Sculpting Algorithm [13]. The result for this example is shown in Fig. 5. From this figure we can infer that the minimum control frequency allowed to sustain transient chaos is to apply a control every $k = 5$ iterations, provided that for $k = 6$ a control bigger than $u_{max} = 0.5$ would be needed. We can see an example of the controlled trajectory in Fig. 6(a), whereas in Fig. 6(b) we can see the control applied, that is

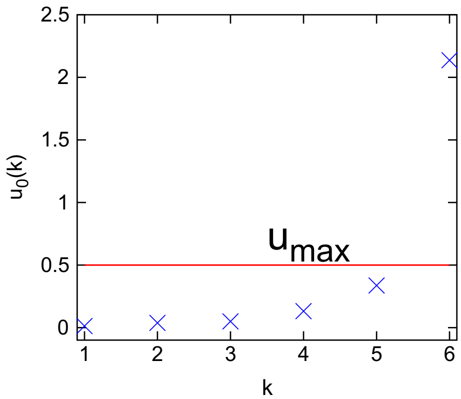

Fig. 5. Numerical estimation of $u_0(k)$, the control needed when applying a control every k iterations ('×'), vs the number of iterations k. We use here $\xi_0 = 0.01$. The value of $u_0(k)$ is determined using the partial control scheme for the corresponding $\xi_0(k)$, that as we know grows with k. The red line indicates the maximum control allowed in the example considered in the text. This implies that the minimum control frequency occurs when we apply the control every $k = 5$ iterations, as long as for $k > 5$ a control bigger than u_{max} would be necessary. (For interpretation of the references to colour in this figure caption, the reader is referred to the web version of this article.)

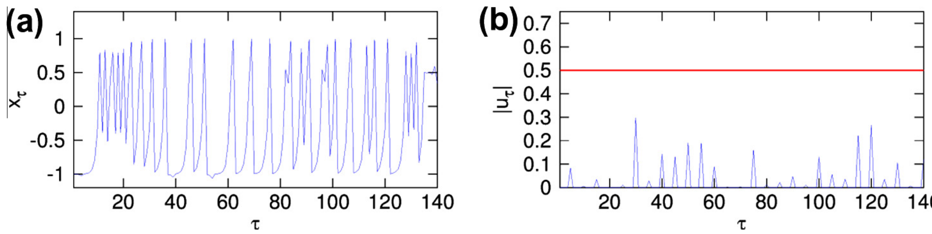

Fig. 6. A controlled trajectory of the tent map with a disturbance $\xi_0 = 0.01$ when applying the control every $k = 5$ iterations (the minimum control frequency) (a) and applied control (b). Note that the applied control is always smaller than the maximum allowed control in this example, $u_{max} = 1$.

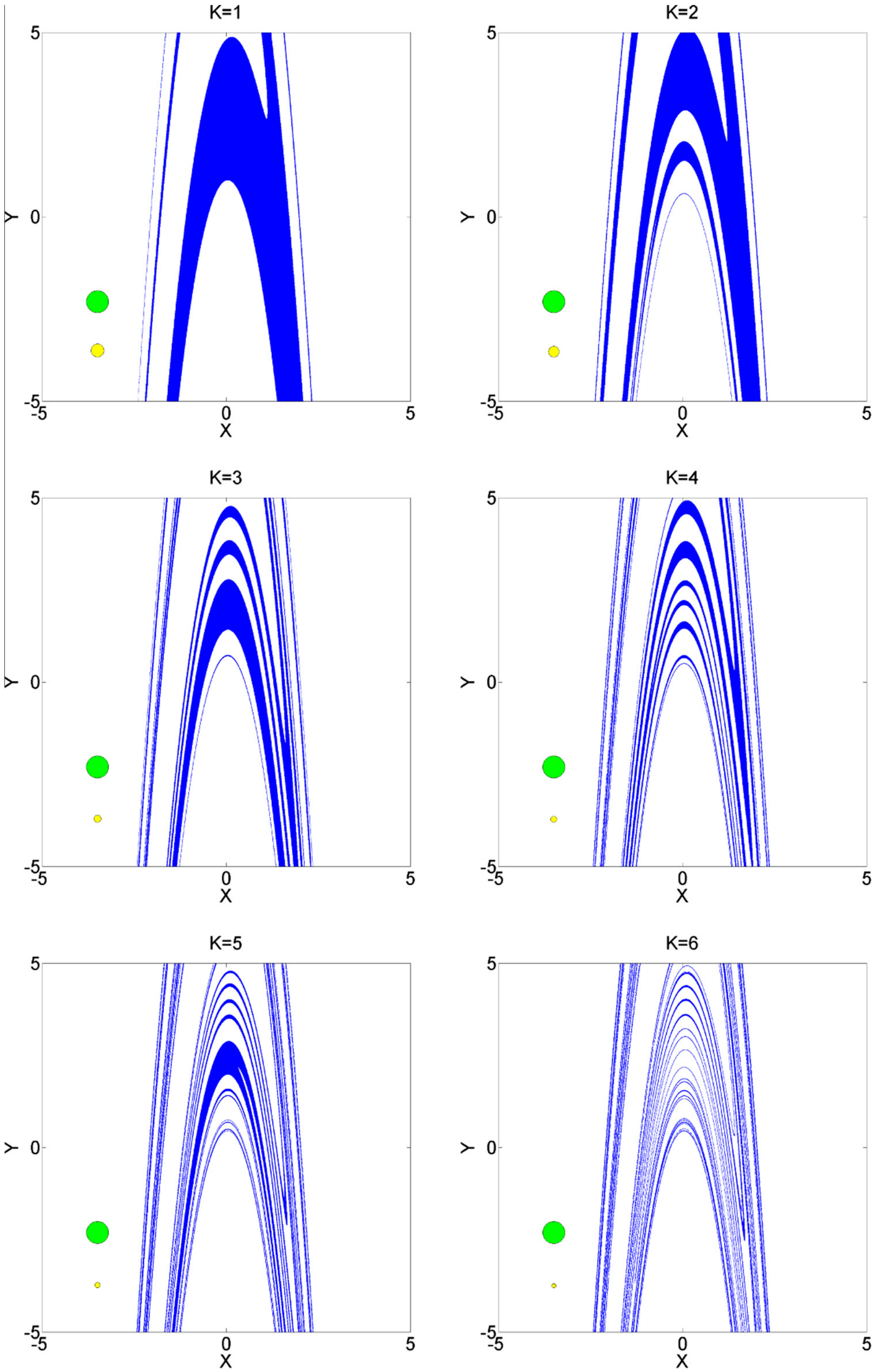

Fig. 7. We show here in blue the safe sets for the Hénon map for $a = 2.16$ for different values of k. The green ball is the maximum admissible disturbance and the yellow ball the maximum admissible control, which clearly decreases with k. For every k we consider a constant disturbance with a maximum value equal to 0.3. (For interpretation of the references to colour in this figure caption, the reader is referred to the web version of this article.)

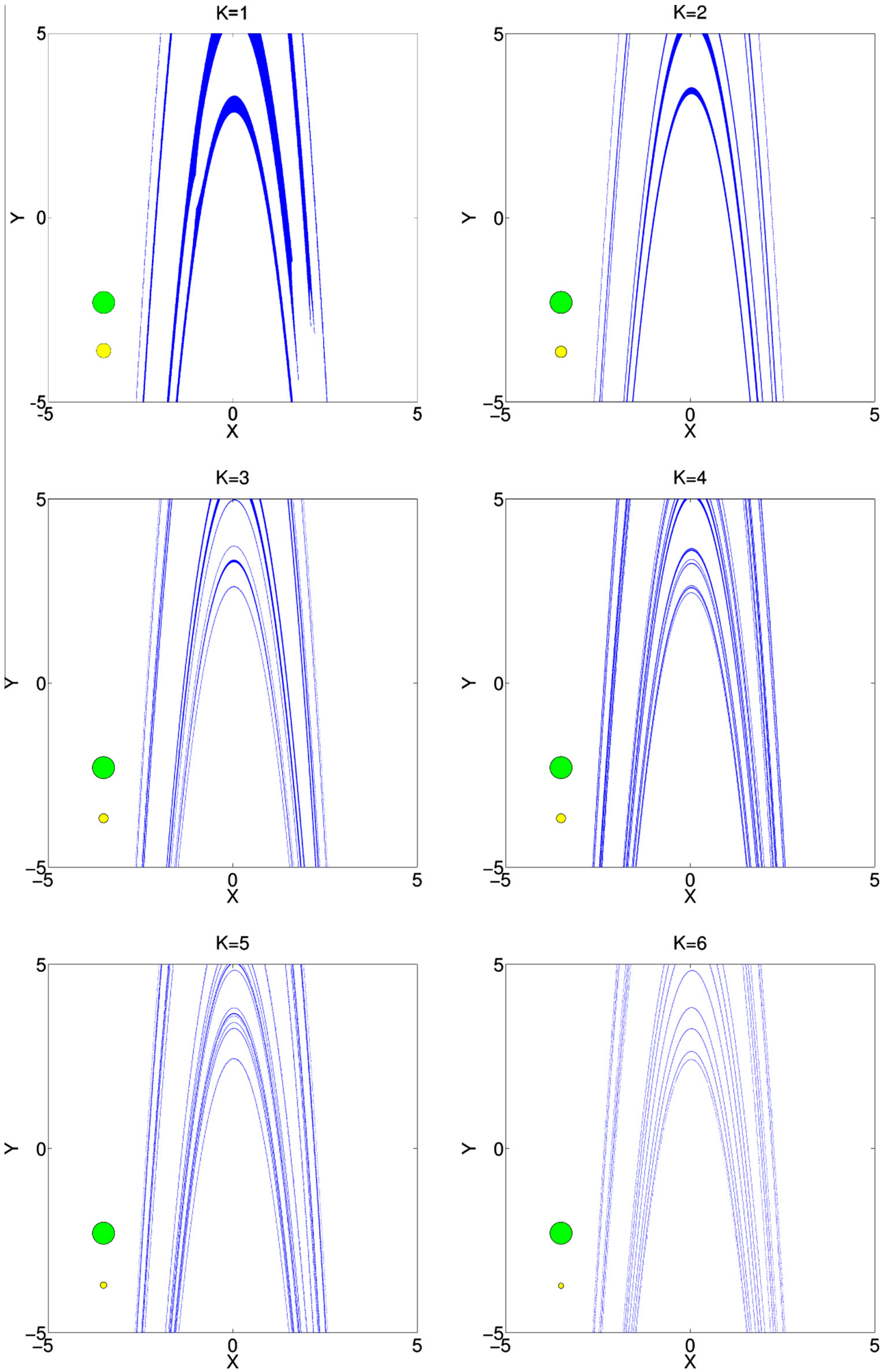

Fig. 8. We show here in blue the safe sets for the Hénon map for $a = 3$ for different values of k. The green ball is the maximum admissible disturbance and the yellow ball the maximum admissible control, which clearly decreases with k. For every k we consider a constant disturbance with a maximum value equal to 0.3. (For interpretation of the references to colour in this figure caption, the reader is referred to the web version of this article.)

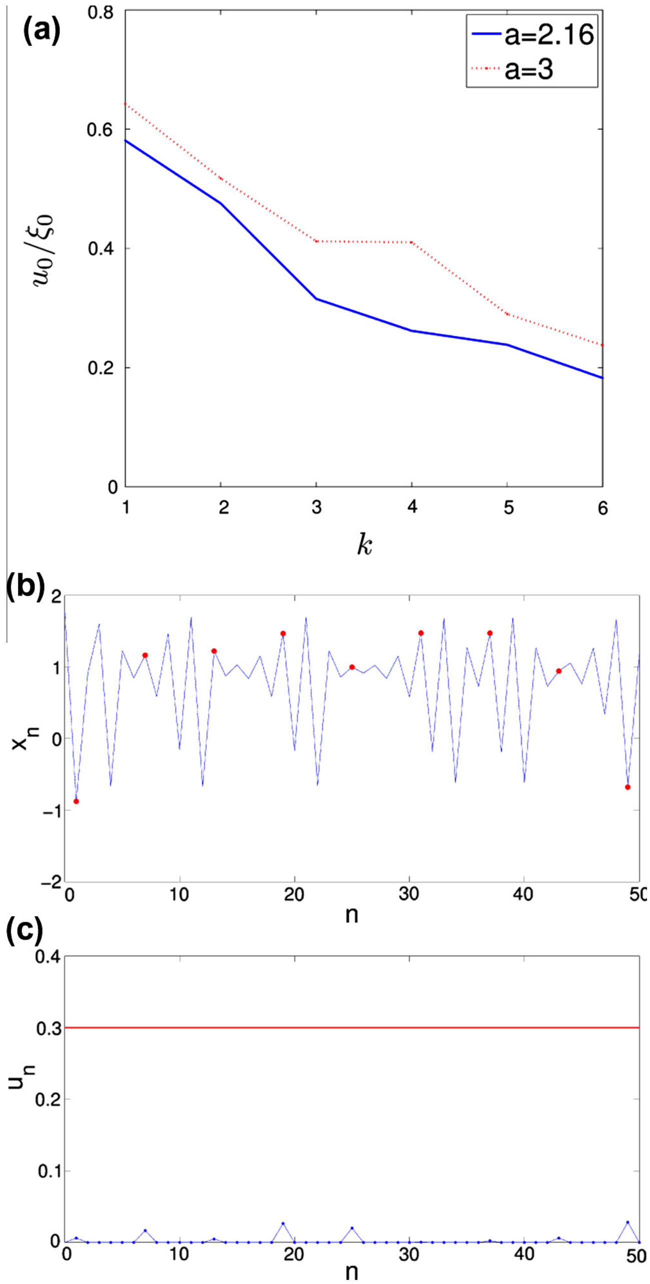

Fig. 9. (a) Ratio $u_0(k)/\xi_0(k)$ computed for constant $\xi_0(k) = 0.3$ for k iterates of the Hénon map with $a = 2.16$ (line) and $a = 3$ (segments). As expected the ratio decreases faster in the case of $a = 2.16$ due to the fact that the size of the escaping region is smaller. (b) Time series of the x variable of a controlled trajectory with $k = 6$ and $\xi_0(6) = 0.3$ and (c) the control applied in this case, which is clearly smaller than the noise value $\xi_0(6) = 0.3$ (shown with a red line). (For interpretation of the references to colour in this figure caption, the reader is referred to the web version of this article.)

clearly smaller than the maximum prescribed value u_{max}. Thus, by using partial control we have found a way to keep the system's trajectories bounded by applying a control smaller than the maximum value allowed and as unfrequently as possible. We note that given u_{max} and by using any other strategy to sustain transient chaos, that would require a bigger $u_0(k)$ given $\xi_0(k)$, a higher control frequency would be required. Thus, our reasoning above provides a way to minimize the control frequency needed to sustain transient chaos.

Note that this procedure is based on the property unveiled for the tent map: that the ratio $u_0(k)/\xi_0(k)$ at a constant value of $\xi_0(k)$ decreases with k. This compensates in part the fact that the disturbance effect would grow with k. In the following section we show that the same property holds for the Hénon map, which suggests that this result is of a general nature and thus our procedure to minimize the control frequency when sustaining transient chaos could be applied to any dynamical system.

5. Results for the Hénon map

In order to investigate the generality of our results, we consider now the Hénon map, defined as

$$\begin{cases} x_{n+1} = a - by_n - x_n^2 \\ y_{n+1} = x_n. \end{cases} \tag{13}$$

If we fix the parameter $b = 0.3$ and we take a value of $a > 2.12$ almost all the initial conditions escape after a finite amount of time from the square $Q = [-5, 5] \times [-5, 5]$. The behavior of the trajectories while they are inside the square $Q = [-5, 5] \times [-5, 5]$ is chaotic, but not permanent. That means that after a finite time of complex behavior inside that square, almost all the deterministic trajectories escape, and in the presence of disturbances all trajectories escape.

The goal here is to check if for this map there is a decrease in the control/disturbances ratio as a function of k for a fixed value of $\xi_0(k)$, that as we have shown above is the key to minimize the control frequency. For that reason, we have computed the safe sets associated for $k = 1, 2, 3, 4, 5$ and 6 iterates of the Hénon map using a constant disturbance amplitude $\xi_0(k) = 0.3$ in all the simulations. For the safe sets computed, we have always looked for the minimum control $u_0(k)$. That means that for every k the safe sets found have been computed with a u_0 below which no safe set exists. The precision in the value of $u_0(k)$ is of three decimals.

The resolution that we have used for the computation of the safe sets has been a challenge since what seemed to be right for $k = 1, 2, 3$ was clearly not enough for $k = 4, 5, 6$. The reason for this phenomenon is that with every k the complexity of each safe set seems to increase considerably. That is why the computations of the safe sets for $k = 1, 2, 3$ have been done with a resolution of 9000×9000 points while those of $k = 4, 5, 6$ with a resolution of 12000×12000. The methodology that we have followed to choose an appropriate resolution was to increase recurrently the resolution used with the Sculpting Algorithm until we got two different resolutions in which the difference between the minimum $u_0(k)$ computed were rather small. Interestingly, for these resolutions the appearance of the safe sets was always almost identical.

In Fig. 7 we show the safe sets that we have computed for the Hénon map with parameters $a = 2.16$ and $b = 0.3$ and constant $\xi_0(k) = 0.3$. For these parameters the state of the system is very close to the boundary crisis that appears when $a = 2.12$, so we expect the rate of escape to be low. An interesting feature that we can see in this figure is that as we increase the value of k the shape of the safe sets becomes more and more complex. But the main result that this figure shows is that at a constant disturbance the control needed to keep the trajectories bounded is severely reduced as is increased k.

We have also done another set of simulations for the Hénon map with a value $a = 3$, that can be considered to be far away from the crisis. We can see the safe sets computed for different k values in this situation again for constant $\xi_0(k) = 0.3$ in Fig. 8. In this case the escape rate is bigger than in the previous example and the trajectories will leave Q much faster. Again, here it is possible to observe that as k is increased the minimum control needed to avoid escapes decreases, but not as fast as for $a = 2.16$. This is confirmed by the segments in Fig. 9(a), where the ratios $u_0(k)/\xi_0(k)$ for fixed $\xi_0(k) = 0.3$ are shown. There we can see that this ratio decreases with k in both cases and that the values are smaller for $a = 2.16$, closer to the crisis, than for $a = 3$, a feature that we also observed for the tent map.

As an example of how the control would work in this case, in Fig. 9(b) we show a time series of x for a controlled trajectory of the Hénon map with $a = 2.16$, where we have chosen to apply control every $k = 6$ iterations and with a disturbance $\xi_0(6) = 0.3$. In Fig. 9(c) we show the control applied: we can see that it is applied every $k = 6$ iterations and it is well below the maximum disturbance $\xi_0(6) = 0.3$, that is highlighted with the red line.

Summarizing, we have seen that the features previously observed for the tent map also apply for the Hénon map, which implies that for this system we could also minimize the control frequency once the value of u_{max} is known.

6. Conclusions and discussion

In this paper we have investigated what is the minimum control frequency needed to sustain transient chaos in a system in presence of disturbances, showing that the partial control method gives a way to minimize such frequency. We have shown that in spite of the fact that the effects of the disturbances grow with the number of iterations, it is possible to minimize the control frequency.This result is possible due to an interesting property that we have derived analytically for the tent map: at constant disturbances, the minimum control/disturbances ratio required, decreases with the number of iterates k towards an asymptotic value. Furthermore, we have shown that this value is smaller as we get closer to the parameter values for which the chaotic saddle arises. We have shown that our results also hold for the Hénon map, so we believe that in principle they would hold for any dynamical system with a chaotic saddle, although an exact proof on the generality of our results for any system would require further theoretical investigation. Our work also shows that the main advantage of the partial control method, its ability to minimize the control needed to sustain transient chaos in presence of disturbances, can also be used (in a more indirect way) to achieve other valuable control goals.

Acknowledgements

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References

[1] Lai Y-C, Tel T. Transient chaos. Complex dynamics in finite time scales. New York: Springer-Verlag; 2011.
[2] Grebogi C, Ott E, Yorke JA. Phys Rev Lett 1982;48:1507.
[3] Tél T. J Phys A: Math Gen 1991;24:L1359.
[4] Schwartz IB, Triandaf I. Phys Rev Lett 1996;77:4740.
[5] Kapitaniak T, Brindley J. Phys Lett A 1998;241:41.
[6] Dhamala M, Lai YC. Phys Rev E 1999;59:1646.
[7] Yang W, Ding M, Mandell A, Ott E. Phys Rev E 1995;51:102.
[8] Shulenburger L, Lai Y-C, Yalcinkaya T, Holt RD. Phys Lett A 1999;260:156.
[9] In V, Spano ML, Ding M. Phys Rev Lett 1998;80:700.
[10] Zambrano S, Sanjuán MAF, Yorke JA. Phys Rev E 2008;77:055201(R).
[11] Zambrano S, Sanjuán MAF. Phys Rev E 2009;79:026217.
[12] Sabuco J, Sanjuán MAF, Yorke JA. Chaos 2012;22:047507.
[13] Sabuco J, Zambrano S, Sanjuán MAF, Yorke JA. Commun Nonlinear Sci Numer Simul 2012;17:4274.
[14] Sabuco J, Zambrano S, Sanjuán MAF. New J Phys 2010;12:113038.
[15] Aguirre J, d'Ovidio F, Sanjuán MAF. Phys Rev E 2004;69:016203.
[16] Smale S. Bull Amer Math Soc 1967;73:747.