Ecological Complexity 19 (2014) 1–8

Contents lists available at ScienceDirect

Ecological Complexity

journal homepage: www.elsevier.com/locate/ecocom

Original Research Article

When less is more: Partial control to avoid extinction of predators in an ecological model

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ARTICLE INFO

Article history:
Received 17 July 2013
Received in revised form 10 February 2014
Accepted 10 February 2014
Available online

Keywords:
Partial control
Transient chaos
Species extinction

ABSTRACT

Extinction of a species is one of the most dramatic processes in ecology. Here we use an extended version of the McCann–Yodzis three-species food chain model proposed by Duarte et al. (2009), where a cooperative hunting term was added to the original McCann–Yodzis model and where the three species coexist: resources, consumers and predators. We consider a situation for which a chaotic transient is present in the dynamics implying the predators extinction. Taking into account that the system is affected by external disturbances, we implement a new control method, the partial control method, with the goal of avoiding the extinction with a control applied smaller than the external disturbances of the system. We have also shown that the partial control method implies smaller controls.

1. Introduction

Models of predator–prey systems constitute an important research field in ecology. Different dynamical behaviors, like periodic orbits or strange attractors are frequently present in these models, suggesting the complexity that the interaction between species may reach. From an ecological point of view, one of the most dramatic events occurs when the populations are driven by the dynamics towards an undesirable state. Overpopulation or species extinction are typical situations that may require expensive efforts in the attempt to control the process. In this sense, a good understanding of the underlying causes constitutes a necessary step previous to design a suitable control strategy.

In this work, we consider a particular dynamical behavior called transient chaos. This phenomenon appears in many systems such as a thermal pulse combustor (In et al., 1997), a periodically driven CO_2 laser (Dangoisse et al., 1986), or a voltage collapse (Dhamala and Lai, 1999). The main cause generating this transient behavior from a topological point of view lies at the presence of a chaotic saddle in the phase space. This topological object arises when a chaotic attractor collides with its own basin boundary producing a transient chaotic behavior of trajectories before eventually escaping towards an external attractor (see Sabuco et al., 2010; Lai and Tél, 2011). Likewise, a wide variety of ecological models have been proposed (Hastings and Higgins, 1994; McCann and Yodzis, 1994; Sinha and Parthasarathy, 1996; Gyllenberg et al., 1996; Vandermeer and Yodzis, 1999; Schreiber, 2001) explaining successfully the process of species extinction, as the consequence of the existence of a boundary crisis in phase space. This significant result suggests the importance to deepen in the knowledge of transient chaotic behavior in the context of ecological complexity.

With the purpose of studying the transient chaos in an ecological model, we have chosen a three-species food chain model proposed by Duarte et al. (2009). This model is based on the McCann–Yodzis model (McCann and Yodzis, 1995), where three species coexist: resources, consumers and predators. The interest of this model relies on the simple and plausible explanation of the problem of species extinction, without the necessity to consider temporal or spatial variations and external factors. In addition, the parameters of this model are ecologically meaningful because they were derived from bioenergetics (McCann and Yodzis, 1995).

Following this model, Duarte et al. (2009) have proposed an extended model with the possibility of predators to cooperate to hunt. This behavior has been found in several different situations such as populations involving mammals (Stander, 1991; Mills, 1978), fishes (Cook and Streams, 1984; Major, 1978), insects (Nakasuji and Dyck, 1984) and spiders (Rypstra, 1985; Ward and Enders, 1985), see also Dugatkin (1997). So far, this behavior has typically been modeled as a cooperation strategy in the context of game theory (Packer and Ruttan, 1988), while the theoretical approaches in the context of nonlinear dynamics are weak. With this motivation, Duarte et al. (2009) have added a simple term in the original McCann–Yodzis model, which involves that some

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http://dx.doi.org/10.1016/j.ecocom.2014.02.002
1476-945X/

individuals cooperate during prey's hunting. This term introduces a small Allee effect in the system, that can be adjusted depending on the different degrees of cooperation, recovering the original McCann–Yodzis model when no cooperation exists.

The dynamics of this extended model presents two different behaviors depending on the values of two parameters. In one case, all species coexist in a chaotic attractor, while in the other, transient chaos appears, involving the extinction of the predators population. Duarte et al. (2009) explore how these two states were related with the degree of cooperation, and they found that the extinction of predators is stimulated by his own cooperation strategy, that is, the increase of cooperation drives the predators to the extinction. This relevant result is consistent with recent studies which suggest the importance of intraspecific competition between predators in the stabilization of the dynamics of the three-species food chain models (Deng, 2006).

In the case that the system falls in the extinction state, the question that naturally arises is the possibility to avoid it, sustaining the dynamics in the transient behavior. Following this idea, Duarte et al. (2009) applied the control method described by Dhamala and Lai (1999). They showed that, in absence of disturbances, the transient chaos can be sustained avoiding the predators extinction. However, all real systems are affected by certain external disturbances, producing large deviations in a nonlinear deterministic system (Weis and Knobloch, 1990). In fact, many control methods that are effective without disturbances, can fail when the disturbances are present.

In recent years, a novel control method called *partial control* has appeared in the literature (see Zambrano et al., 2008; Zambrano and Sanjuán, 2009; Sabuco et al., 2012a,b). This control method is applied in situations where transient chaos is present and the system is subjected to external disturbances. The main result of this paper is the successful control of trajectories in the ecological model introduced in Duarte et al. (2009). Furthermore, we show that the amount of control needed is even smaller than in other control strategies.

The structure of the paper is as follows. Section 2 is devoted to the description of the ecological model. The main ideas of the partial control method and its application to the model is described in Section 3. And in Section 4 the safe sets are computed. A comparative analysis of the partial control method with the one used in Duarte et al. (2009), is given in Section 5, where we show that the amount of control needed by using the partial control method for a given amount of noise might be much smaller than in other control methods. Finally, some conclusions are drawn in Section 6.

2. Description of the ecological model

We use a three species food chain model proposed by Duarte et al. (2009). This model is an extension of the McCann–Yodzis model, which describes the dynamics of the population density of a resource species R, a consumer C and a predator P. In addition, Duarte et al. (2009) propose the introduction of a nonlinear term with the aim to model the possibility of the predators to cooperate to hunt. The resulting model is given by the following set of nonlinear differential equations:

$$\begin{aligned} \frac{dR}{dt} &= R\left(1-\frac{R}{K}\right) - \frac{x_c y_c C R}{R+R_0} \\ \frac{dC}{dt} &= x_c C\left(\frac{y_c R}{R+R_0} - 1\right) - \psi(P)\frac{y_p C}{C+C_0} \\ \frac{dP}{dt} &= \psi(P)\frac{y_p C}{C+C_0} - x_p P. \end{aligned} \tag{1}$$

The biological assumptions of this model are: (i) continuous growth and overlapping generations are allowed for each species. (ii) The resource population grows logistically. (iii) Consumers and predators dies off exponentially without food. (iv) The feeding rate of consumers and predators saturates at high food levels. Its is important to point out that we are assuming populations big enough to dismiss lattice effects (Henson et al., 2001) which in the case of small populations could change dramatically the dynamics and therefore it should be considered in any control method.

Following McCann and Yodzis (1995), we fix the parameters as: $x_c = 0.4$, $y_c = 2.009$, $x_p = x_i = 0.08$, $y_p = 2.876$, $R_0 = 0.16129$ and $C_0 = 0.5$. The term $\psi(P) = x_p(1-\sigma)P + x_i\sigma P^2$ in the equations represents the reproduction kinetics of predators. In this term Duarte et al. (2009) included the parameter $\sigma \in [0, 1]$, which reflects the fraction of predators that cooperate to hunt. Note that the McCann–Yodzis model is a particular case of this model when $\sigma = 0$.

The interest of this model lies in the fact that the dynamics presents transient chaos depending on the values of the carrying capacity K and the degree of cooperation σ. Analyzing the nonlinear dynamics of the system, it is possible to find the different pair of values (K, σ) for which the boundary crisis takes place. For instance, fixing $K = 0.99$, the boundary crisis appears at $\sigma_c = 0.04166$. This critical value separates the two different dynamical regions. Before the crisis, for $\sigma < \sigma_c$, two attractors coexist in phase space: one chaotic attractor where all the species coexist, and one limit cycle where no predators exist (see Fig. 1a). However, after the crisis (Fig. 1b), the only asymptotic attractor is the limit cycle where no predators exist. Such a crisis becomes the limit cycle in the global attractor. Therefore a trajectory close to the chaotic saddle follows the typical time series represented in Fig. 1c, where initially the predators population has a chaotic transient and then it collapses to zero becoming extinct.

Under these conditions, it is clear that in the absence of an external action, the population of predators is doomed to extinction. In this sense, a question reaches naturally: is it possible to avoid the extinction? Obviously, the answer depends on the realistic possibility to perturb the system to sustain the transient chaos behavior. One approach might be to decrease the resource carrying capacity K or the degree of cooperation between predators σ. However, from an ecological point of view, to change these parameters is not always accessible or takes up a large amount of time acting over the system. Nevertheless, it is possible to modify the dynamics, acting directly on a given dynamical variable. In this sense, we can find in the literature a wide number of control methods, that in different ways, deal with the same problem. We find methods like the "target oriented control" or "proportional feedback" that are able to stabilize the unstable or even the chaotic dynamics around an asymptotic stable equilibrium and have been successfully applied to ecological models (Carmona and Franco, 2011); Dattani et al., 2011; Franco and Peran, 2013). Other strategies, like the "adaptive limiter control", reduce the fluctuations of the populations, with the aim of bounding the dynamics within a certain range and therefore avoiding undesirable or dangerous escapes (Franco and Hilker, 2013). In particular this method preserves the chaotic dynamics, a feature that we consider highly desirable in order to maintain the natural evolution of the ecological system. Finally, we have the method described in Dhamala and Lai (1999) that has also been applied in Duarte et al. (2009), attempting to avoid the extinctions identifying a escape region and not allowing the system to enter inside. Surprisingly, in the literature it is very difficult to find control methods applied to systems with noise, which we consider that is something rather common in all real systems, where the noise will enclose all the uncertainty about the dynamics of the system, like modeling mismatches or external disturbances. In addition, we consider that it would be very helpful to have a method being able to fix an upper control bound that ensures the control of the system regardless of

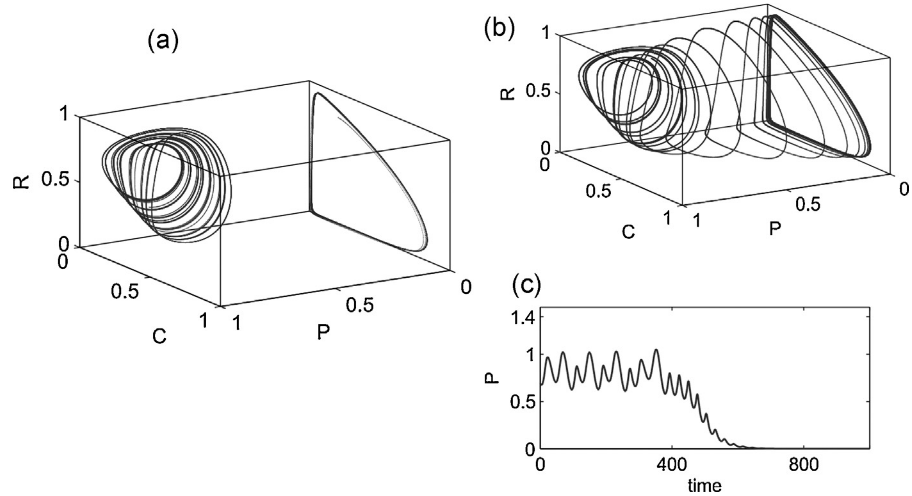

Fig. 1. Dynamics of the extended McCann–Yodzis model proposed by Duarte et al. (2009) from Eq. (1). Depending on the values of the parameters (K, σ) different dynamics are possible. Fixing $K = 0.99$, the boundary crisis appears at $\sigma_c = 0.04166$. (a) Before the boundary crisis ($K = 0.99$, $\sigma = 0$), there are two possible attractors depending on the initial conditions: one chaotic attractor where the three species coexist, and one limit cycle where only the resources and consumers coexist. (b) After the boundary crisis ($K = 0.99$, $\sigma = 0.07$), the limit cycle is the only asymptotic attractor. (c) Time series of the predators population corresponding to the case (b), where the chaotic transient before the extinction is shown.

the system evolution. With this motivation a new control method is introduced in the next section. By acting just over the population of predators during discrete times, we will show how this method is able to sustain the dynamics in the transient chaos regime.

3. Application of the partial control method

The partial control method has been successfully applied to several paradigmatic dynamical systems, such as the Hénon map and the time-2π map associated to the Duffing oscillator (Sabuco et al., 2012a,b) for a parameter choice where the trajectories escape outside a certain region Q in phase space. In particular, when we consider the dynamics after a boundary crisis, the system possesses a transient chaotic behavior in a bounded region in phase space, previous to a situation in which the trajectory escapes towards an attractor outside this region. When the dynamics is affected by noise, somehow it might help the trajectory to escape from the region earlier. The goal of the partial control is to apply a control in order to avoid the escape of the trajectory from this region Q, and what is surprising is that the amount of control we need is smaller than the external disturbance acting on the dynamical system.

The presence of a boundary crisis in the ecological model, as described in the previous section, and the requirement of a discrete control, makes it especially suitable the application of the partial control method to avoid the extinction of predators in this system. To implement it, we need a map and to define a region Q in phase space, where we want to sustain the dynamics. The complete dynamics in presence of an external disturbance ξ_n and after the application of a control u_n is described by the iterative equation $q_{n+1} = f(q_n) + \xi_n + u_n$. The only assumption we consider on the disturbances and control is to be bounded, that is, $|\xi_n| \leq \xi_0$ and $|u_n| \leq u_0$, and when this happens we say that we have admissible disturbances and controls. A point $q \in Q$ is considered *safe*, if the next iteration of this point $f(q)$ under the action of the map and affected by the external disturbance falls inside Q once a control $|u_n| \leq u_0 < \xi_0$ is applied. We can say that under the previous considerations, a safe point is controlled and consequently remains in Q with an applied control smaller than the disturbance. The set of all safe points in Q is called the *safe set*. And there is an algorithm called *Sculpting Algorithm* (Sabuco et al., 2012b), that computes automatically the safe set given a map, a region Q in phase space and admissible disturbances and controls. So that, our goal here is to compute the safe set for the ecological model described in the previous section. The Sculpting Algorithm works in such a way that it rejects, in the first iteration, the points q_n for which $q_{n+1} = f(q_n) + \xi_n$ need a control $|u_n| > u_0$ to get back to the region Q. The points that survive are a subset of Q, and the process is repeated until it finally converges. As a result, we obtain the safe set containing all safe points, which provide those points that are controlled with admissible disturbances and controls.

4. Finding the safe sets and controlled trajectories

Our goal now is to find the safe sets for our ecological model. We have chosen the parameter values $K = 0.99$ and $\sigma = 0.07$, that corresponds to the region after the boundary crisis, since we are interested in the transient chaotic regime.

In order to apply the Sculpting Algorithm to find the safe sets, we need as basic ingredients a map, a region Q in phase space where the map is acting and where we want the dynamics to stay in, and an admissible choice of disturbances and controls. As a consequence, the first step is to obtain such a map. In our case, a one-dimensional map can be obtained from the three-dimensional flow, by computing the local minima of the time series of $P(t)$. As a result, we obtain a set of points of the form (R_n, C_n, P_n), where R_n and C_n are computed at the precise moment for which the $P(t)$ time series has a local minimum. This set of points generates an approximately one-dimensional curve in phase space as shown as a green line in Fig. 2.

In addition, the set of local minima is practically parallel to the P-axis, where R_n and C_n are practically constant. This important feature allows us to apply the control only to P_n. At this point, we have to notice that we will not always be lucky in this sense, and the potential to reduce the variables to control, will depend on the particular structure of the chaotic saddle and the variable/s that we will want to control.

Next, we build a one-dimensional map with the successive local minima (P_n, P_{n+1}), where P_n denotes the nth local minimum. Notice that this is not a discretization on time and therefore the return times between two consecutive minima is not the same. In our case we obtain typical return times within the range of 25–50 time units. Since a particular initial condition yields only a few pair of points (P_n, P_{n+1}), we simulate a large number of initial conditions to

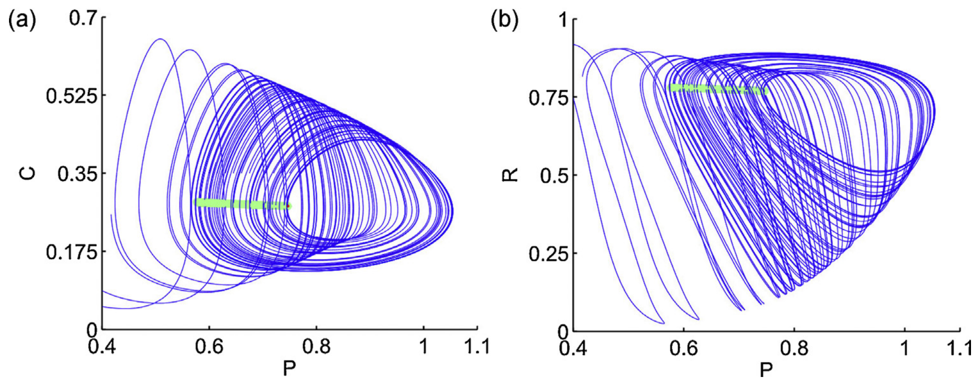

Fig. 2. Set of local minima in phase space. Blue line: trajectories in phase plane (P, C) and (P, R). Green line: set of local minima of $P(t)$ used to build the one-dimensional map. As shown, the set of local minima is approximately parallel to the P-axis. (For interpretation of the references to color in this legend, the reader is referred to the web version of the article.)

obtain a big data set of the iterated map. At this point, we could use this set of points directly to compute the safe set by the following method. Due to the fact that we need to evaluate the image of P_n, we could computed the expected image $f(P_n)$ taking, for example, the mean of the images of the nearest points belonging to the data set. However this process is computationally expensive and therefore we perform an accurate polynomial fitting to obtain the analytical dependency $P_{n+1} = f(P_n)$, making easier the computation of the safe set.

In Fig. 3, we show the polynomial fit obtained from the set of local minima of the time series of $P(t)$. Notice that, the iterates of any initial point for which $P_n > P^*$, follow a chaotic dynamics until they finally asymptotes to zero when it crosses a critical value $P_n < P^*$, which actually implies the extinction of the predators population.

Now, we proceed to introduce an upper bound ξ_0 for the disturbances in the return map. The fact that we only apply the disturbance to the P_n variable and not to R_n or C_n might seem restrictive, but we have to take into account that the local minima of $P(t)$ is placed along the unstable manifold of the chaotic saddle (see Fig. 2). Therefore, the trajectories affected by noise (for instance, with a continuous noise) mainly spread out along this manifold, or in our case, along the P_n values. For this reason, to add a disturbance only to the map $P_{n+1} = f(P_n)$ is a very good approximation. In our numerical simulations, we have chosen a uniform noise distribution bounded by ξ_0. Each time that the trajectory crosses the set of the local minima, we insert a disturbance $|\xi_n| \leq \xi_0$.

In the next step, we have to define the initial region Q in phase space where we want to maintain the dynamics of the system. In our case, we want to sustain the dynamics close to the chaotic attractor, avoiding the escape produced when $P_n < P^* = 0.589$, therefore we choose the initial Q to be the interval $P_n \in [0.589, 0.84]$ indicated in Fig. 4. It is important to note that, a suitable initial region Q, must contain the chaotic saddle, which is the responsible for the existence of the chaotic transient.

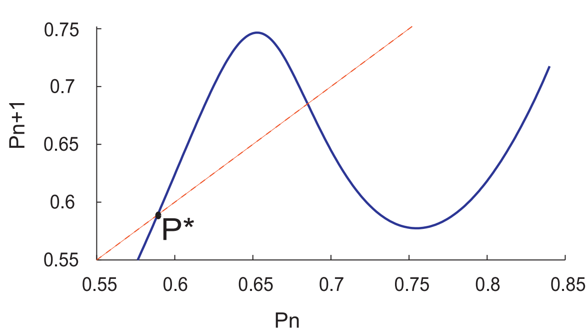

Fig. 3. Return map obtained by using the successive local minima of the time series $P(t)$ as shown in Fig. 2(c). Notice that below $P^* = 0.589$ the trajectory asymptotes to zero. This is the map $P_{n+1} = f(P_n)$ that we use to apply the partial control method.

Once we have defined what do we want to control, we use the Sculpting Algorithm (Sabuco et al., 2012b) in order to find the safe set. The computation of the safe set depends on the chosen values of ξ_0 and u_0 and our observations indicate that for a given ξ_0, we may obtain different safe sets which correspond to different values of u_0. As a matter of fact, the smaller the u_0, the smaller the final safe set. Nevertheless, there is a critical value of u_0 below which no safe set exists. In this case, we have chosen for our simulations $\xi_0 = 0.0114$ and $u_0 = 0.0076$, where u_0 is very close to the minimum value for which the safe set exists. In Fig. 4 we represent the steps of the algorithm to build the safe set.

In Fig. 5 we represent the obtained final safe set composed by the different subsets shown there. We use this safe set to choose the points in phase space that can be controlled as we have described. When the trajectory crosses the set of the minima, we evaluate the value of $f(P_n) + \xi_n$, if the point is inside a safe set we do not apply the control, and if it is outside, we relocate it inside the nearest safe point, resulting the new safe point $P_{n+1} = f(P_n) + \xi_n + u_n$. The criterion to control the point to the nearest safe set is only an option, since in most cases there are other possible points belonging to the safe set which we can reach without exceeding the upper bound of control. From an ecological point of view this flexibility allows us to choose the better option considering our specific needs. For example, depending on our ease to stocking or

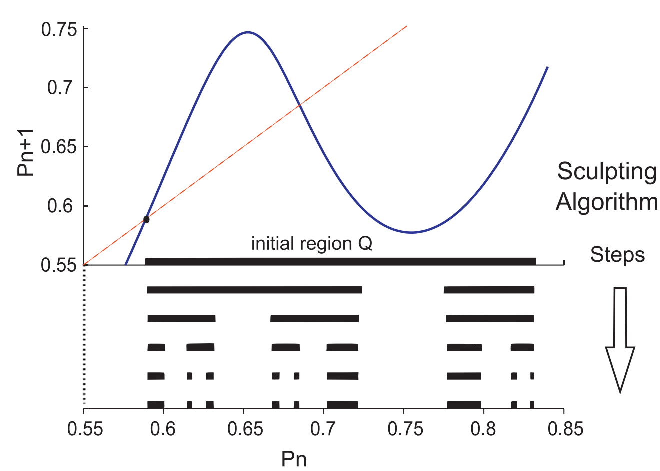

Fig. 4. Steps of the Sculpting Algorithm that converges to the final safe set. The upper bound disturbance and control used by the algorithm are $\xi_0 = 0.0114$ and $u_0 = 0.0076$ respectively. The horizontal *black bars* helps us to visualize the process and represent the points P_n that satisfy the condition to be a safe point at each step.

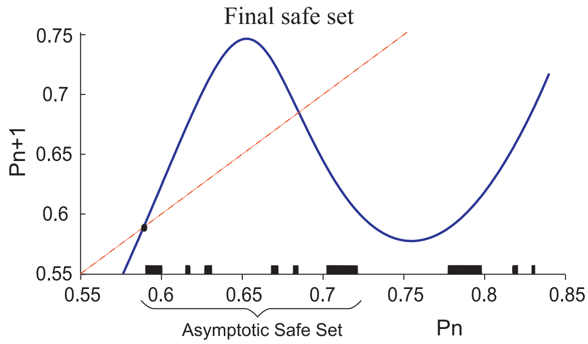

Fig. 5. Final safe set composed of different subsets obtained with the Sculpting Algorithm using $\xi_0 = 0.0114$ and $u_0 = 0.0076$. We also indicate the group of subsets where the dynamics remains trapped, that is, the asymptotic safe set (Sabuco et al., 2012a).

harvesting individuals we can make the choice which involves the smallest effort.

When carrying out the numerical simulations, we have noticed that the dynamics after some iterations do not visit certain regions of the safe set. This subset of the safe set where the trajectories asymptote after the control is applied is called the *asymptotic safe set* (see Fig. 5). For a detailed analysis about the features and direct computation of the asymptotic safe set (see Sabuco et al., 2012a). Controlled trajectories in phase space are shown in Fig. 6, where we also indicate the safe set used with the projections on the set of the minima for a clear visualization. In Fig. 7, we represent the

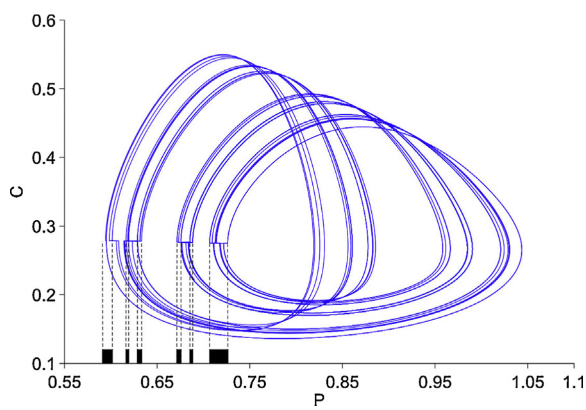

Fig. 6. Controlled trajectory with $\xi_0 = 0.0114$, represented in the phase plane (P, C). We also show the asymptotic safe set calculated with $\xi_0 = 0.0114$ and $u_0 = 0.0076$, and its projection in the set of the minima (dashed line) where the control is applied.

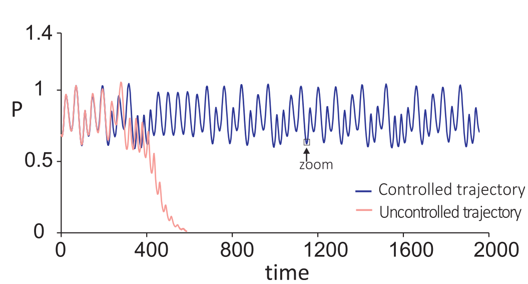

Fig. 7. Red line: time series of the predators population without control exhibiting a escape towards zero, what implies the extinction of the predators. Blue line: controlled time series of the predators population where the extinction is avoided. This time series corresponds to 50 iterations in the return map $P_{n+1} = f(P_n)$. A zoom of one of the minima of the time series of $P(t)$ is shown in Fig. 9 in order to see how the noise and the control are applied. (For interpretation of the references to color in this legend, the reader is referred to the web version of the article.)

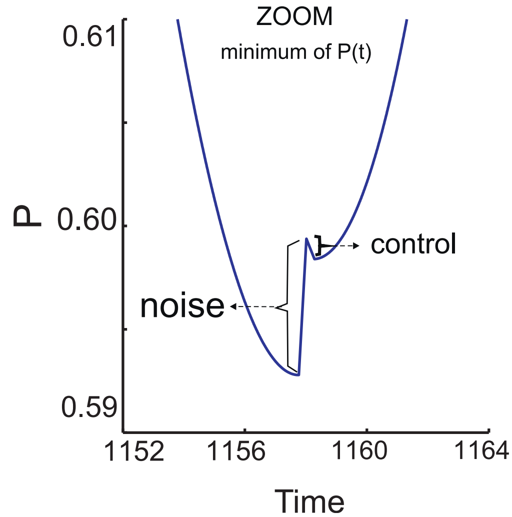

Fig. 8. Zoom of the correction in the local minimum of $P(t)$ indicated in Fig. 8. We can see the noise introduced and the corresponding control applied.

corresponding controlled time series of the predators population (blue line) in contrast to the uncontrolled trajectory (red line), involving the extinction. To visualize how the disturbances and the respective controls are added in a minimum at each iteration, we show in Fig. 8 a zoom of one of the minima of the time series of $P(t)$.

In order to see a further analysis, in Fig. 9 we represent the strength of the noise and the strength of the control applied for 30,000 iterations corresponding to the time interval $[0, 1.2 \times 10^6]$ in the time series of $P(t)$. While the strength of the noise is distributed uniformly between the values 0 and $|\xi_0| = 0.0114$, all values of the control are located under the maximum $|u_0| = 0.0076$, showing that the partial control method works as we expected. We also compare the mean of the noise and the control, obtaining an average control 0.0018 which is less than half the average noise 0.0057. As we will show later, the small average control that we need to use, is another remarkable feature of this control method.

In order to see the adaptability and robustness of the partial control, we compute the safe sets for different values of the upper bound noise ξ_0 in the range $[0.001, 0.057]$. We show the controlled trajectories in phase space in Fig. 10 with the respective safe sets used.

As our results show, the partial control works rather well for different values of ξ_0. The trajectories are sustained in the chaotic region with $|u_n| \leq u_0$ for all the iterations, moreover, as indicated in

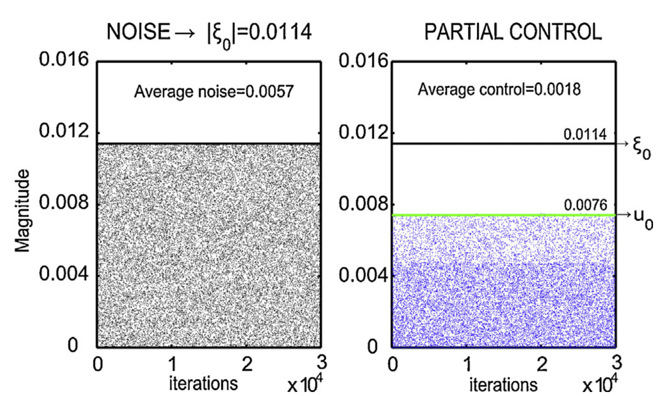

Fig. 9. Strengths of noise and control applied for 30,000 iterations, represented as points instead of bars for a clear visualization. On the left, the strength of the noise that affects the map. On the right, the respective strength of controls applied for the partial control method. We also indicate the upper bound of the noise $\xi_0 = 0.0114$ and upper bound of the control $u_0 = 0.0076$ used to compute the safe set.

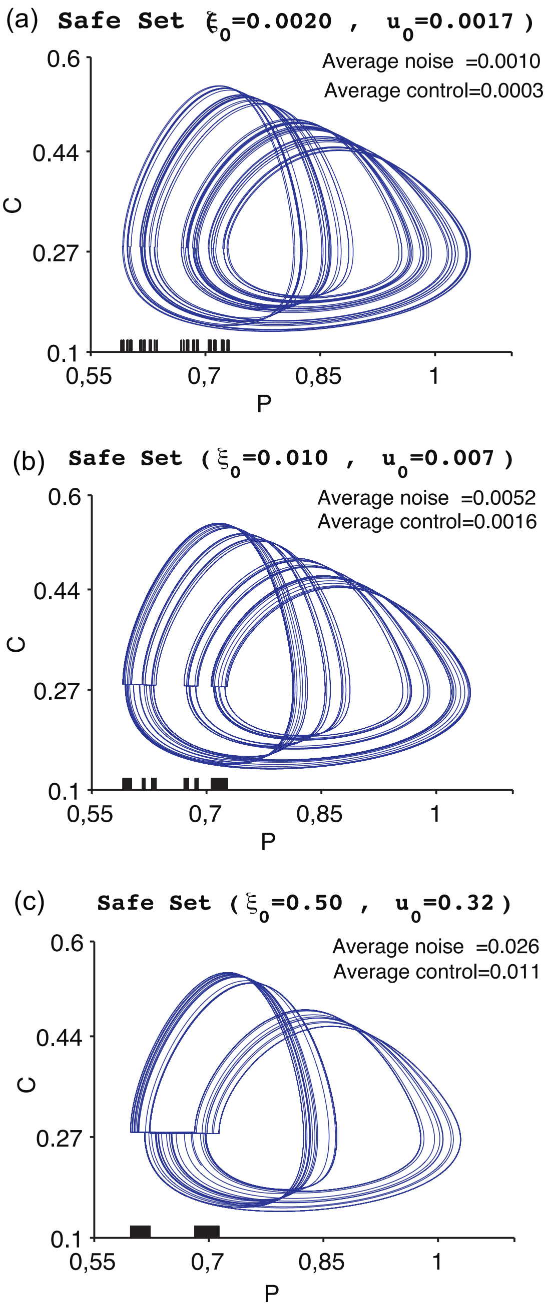

Fig. 10. Controlled trajectories in the phase plane (P, C) and the respective asymptotic safe sets for different noise intensities in the range $\xi_0 = [0.002, 0.50]$.

each case, the average control used is less than twice the noise average. In addition, this method is able to control the system without great modifications of the original dynamics, keeping the chaotic behavior, as a permanent state.

5. The partial control method implies smaller controls

In the paper by Duarte et al. (2009), the authors apply a control method described in Dhamala and Lai (1999), in order to sustain the chaotic behavior after the crisis for the McCann–Yodzis ecological model with cooperative hunting. They do not consider at all any external disturbance, that is, the system they consider is only deterministic.

The main idea here is to consider the problem analyzed by Duarte et al. (2009), including an external disturbance. As a matter

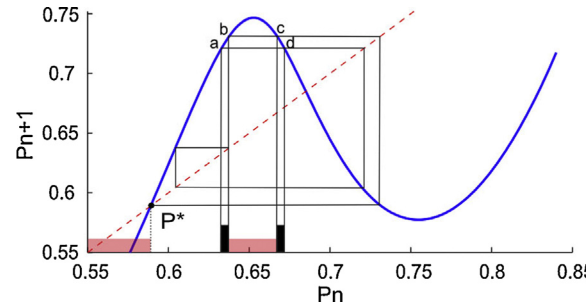

Fig. 11. The *black sets* represent the two target regions used to control the trajectories. The *red sets* represent the escape regions. We colored the regions with different thicknesses to help us in the visualization. (For interpretation of the references to color in this legend, the reader is referred to the web version of the article.)

of fact, we consider the same external disturbance that we have used when we have applied the partial control method in the previous section. And then, we proceed to compare the two control methods.

The method they use is based on the observation that a point $P < P^*$ in the return map, goes quickly to zero. In this sense, it is possible to identify one escaping zone and two target regions, by computing certain preimages of the fixed point P^* as shown in Fig. 11. The idea is that all points inside the escape region, fall above P^* in the next iteration, while the points of the target region, survive a long time before escaping.

The control is defined as follows: we have two target regions (*black sets*) defined by the intervals $[a_1, b_1]$ and $[b_1, c_1]$. Now we define the escaping regions (*red sets*), composed by the points $P < P^*$ and the points between the target regions, see Fig. 11. When a given iteration falls into the escaping region, we apply a control to relocate P inside the nearest target point.

It is necessary to point out that the original escaping region in the deterministic system is just the central gap between the two target regions. However, in the system affected by disturbances, the trajectory can escape directly without passing through this central gap and therefore we also need to control these points.

In Figs. 12, 13 and 14, we represent the control applied in both methods for different values of the strength of the disturbance in the range $\xi_0 = [10^{-3}, 10^{-1}]$. The left plot shows the strength of the noise introduced in the map during 30,000 iterations, and the central and right plot represents the control used in Duarte et al. (2009) and in the partial control method, respectively. We also indicate the average controls applied in both methods.

The first difference between both control methods is the distribution of the controls. While the partial control always use tiny controls, the other method needs sometimes very large controls. In addition, the partial control method uses always an average control much smaller than the average noise, whereas the other method uses a large control in comparison, especially in the case of small disturbances. We can explain these results, taking into account that the control set used in Duarte et al. (2009), is calculated regardless the noise, that is, the control set is the same for any noise, while the safe sets in the partial control, are obtained by taking into account the amount of noise present in the system. As we have seen in the previous section, for small and medium noises, the scale of safe sets is composed by many small pieces, allowing us to control the system with tiny corrections, while the control method used in Duarte et al. (2009) is forced to wait until the trajectory enters into the escape region to apply the control.

In view of the results shown in Fig. 15, we can say that the control strategy used in Duarte et al. (2009) implies larger controls than in the partial control. The reason is that the partial control method is able to anticipate the escape of a trajectory earlier than

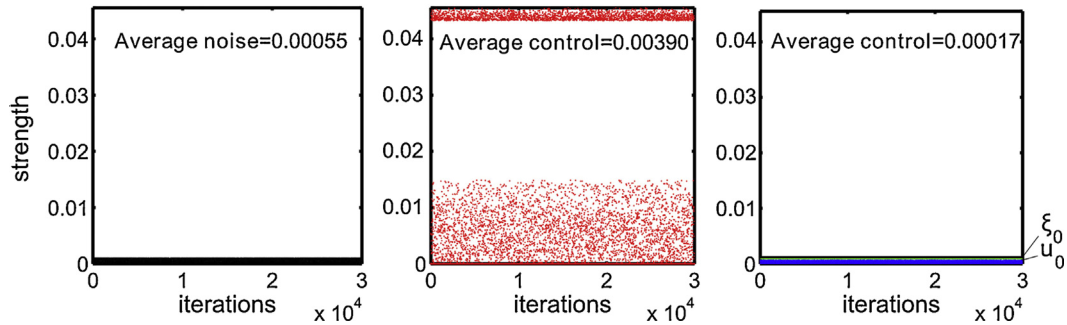

Fig. 12. Control applied by both methods under the same noise condition. On the left, the black points represent the strength of the noise corresponding to a uniform distribution of $\xi_0 = 10^{-3}$. The red points in the center correspond to the strengths of the control method used in Duarte et al. (2009). On the right, the blue points represent the strengths of the control used with the partial control method. We also include the values ξ_0 (*black line*) and u_0 (*green line*) used to calculate the respective safe sets. (For interpretation of the references to color in this legend, the reader is referred to the web version of the article.)

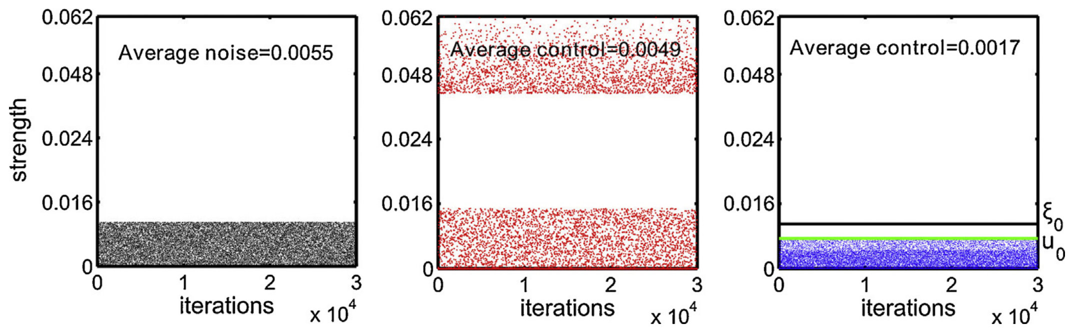

Fig. 13. Same figure as the previous one computed with $\xi_0 = 10^{-2}$.

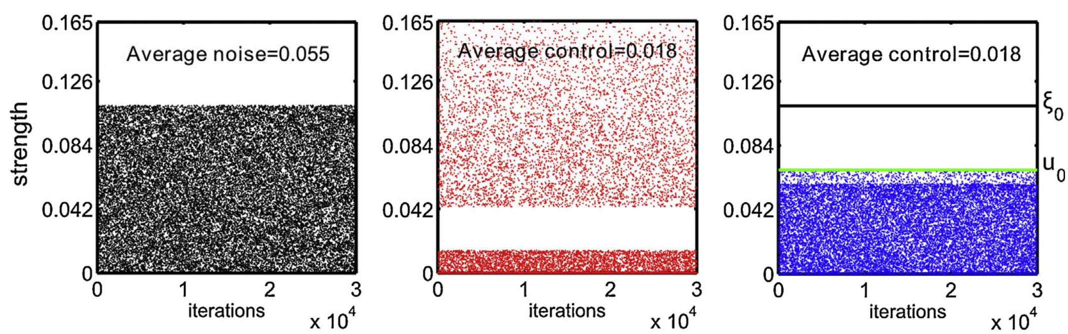

Fig. 14. Same figure as the previous one computed with $\xi_0 = 10^{-1}$.

the other control method, waiting for the last iteration to apply the correction. The natural question is: what is better, the application of small and frequent controls like the ones used in the partial control method, or the large and less frequent controls used in Duarte et al. (2009)?. If the amount of control is the major concern, the Fig. 15 shows how the partial control uses a smaller average control. However if a higher number of interventions involves a higher cost, we can use the safe set computed for $f^2(P_n)$ or $f^3(P_n)$ to reduce the frequency of interventions to a half or a third, at the expense of having larger controls (Zambrano et al., 2014). This flexibility allows the ecologist to choose the best way to achieve his goals, depending on his specific needs. Finally, in the case of large disturbances, the dispersion modifies the chaotic saddle so much, that only a gross control is able to avoid the escape. This is the reason why both methods achieve similar results for larger values of noise.

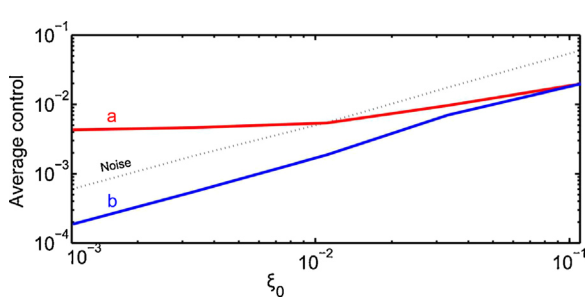

Fig. 15. Comparative of the average controls in a log–log scale. Dashed black line: average strength of the noise in the range $\xi_0 = [10^{-3}, 10^{-1}]$. (a) Red line: average strength of the control method used in Duarte et al. (2009). (b) Blue line: average strength of the partial control. (For interpretation of the references to color in this legend, the reader is referred to the web version of the article.)

6. Conclusions

To avoid the species extinction which time evolution depends on the population dynamics of other species, might be a big challenge from an ecological point of view. Nonlinear interactions among species often result in a complex global dynamics, making that the application of external actions over the system, do not have easy predictable consequences. Moreover, if we introduce an external disturbance in order to reproduce a real system, the task to design a suitable control strategy can be a very difficult issue, which requires a deep knowledge of the complex dynamics.

In this work, we analyze the problem of the species extinction using an extended McCann–Yodzis ecological model affected by external disturbances. This system, which is composed of three species, presents a boundary crisis involving the extinction of one of them. With the aim of avoiding the extinction, we have applied the partial control method, which is a novel successful control method where a control smaller than noise is needed. We show that the method is able to control the dynamics of the three-dimensional flow, applying control just on the predators population. Different strengths of noise have been added, showing in all cases, that the partial control method, was able to sustain the transient chaotic dynamics, avoiding the extinction.

In addition, we show the main features of this method. First, the control is applied in discrete times and the partial control method ensures an upper bound control value, that is, all controls used are placed below this value. In addition, the amount of control applied, is smaller than the given amount of noise. Furthermore, the amount of control is smaller than the control used by other control methods in the literature. The method does not need to have big modifications of the original dynamics, keeping the chaotic behavior. For all these reasons the partial control method reveals a great potential to be applied in real situations affected by noise, where a boundary crisis leads the dynamics to a undesirable state of the system, providing an automatic and minimally invasive strategy to avoid it.

Acknowledgments

Financial support from the Spanish Ministry of Science and Innovation under Project No. FIS2009-09898 is acknowledged.

References

Carmona, P., Franco, D., 2011. Control of chaotic behaviour and prevention of extinction using constant proportional feedback. Nonlinear Anal. Real World Appl. 12, 3719–3726.

Cook, W.I., Streams, F.A., 1984. Fish predation on Notonecta (Hemiptera): relationship between prey risk and habitat utilization. Oecologia 64, 177–183.

Dangoisse, D., Glorieux, P., Hannequin, D., 1986. Laser chaotic attractors in crisis. Phys. Rev. Lett. 57, 2657–2660.

Dattani, J., Blake, J.C.H., Hilker, F.M., 2011. Target-oriented chaos control. Phys. Lett. A 375, 3986–3992.

Deng, B., 2006. Equilibriumizing all food chain chaos through reproductive efficiency. Chaos 16, 043125.

Dhamala, M., Lai, Y.C., 1999. Controlling transient chaos in deterministic flows with applications to electrical power systems and ecology. Phys. Rev. E 59, 1646–1655.

Duarte, J., Januário, C., Martins, N., Sardanyés, J., 2009. Chaos and crises in a model for cooperative hunting: a symbolic dynamics approach. Chaos 58, 863–883.

Dugatkin, L.A., 1997. Cooperation Among Animals: An Evolutionary Perspective, first ed. Oxford University Press, New York.

Franco, D., Hilker, F.M., 2013. Adaptive limiter control of unimodal population maps. J. Theor. Biol. 337, 161–173.

Franco, D., Peran, J., 2013. Stabilization of population dynamics via threshold harvesting strategies. Ecol. Complex. 14, 85–94.

Gyllenberg, M., Osipov, A.V., Söderbacka, G., 1996. Bifurcation analysis of a metapopulation model with sources and sinks. J. Nonlinear Sci. 6, 329–366.

Hastings, A., Higgins, K., 1994. Persistence of transients in spatially structured ecological models. Science 263, 1133–1136.

Henson, S.M., Constantino, R.F., Cushing, J.M., Desharnais, R.A., Dennis, B., King, A.A., 2001. Lattice effects observed in chaotic dynamics of experimental populations. Science 294, 602–605.

In, V., Spano, M.L., Neff, J.D., Ditto, W.L., Daw, C.S., Edwards, K.D., Nguyen, K., 1997. Maintenance of chaos in a computational model of a thermal pulse combustor. Chaos 7, 605–613.

Lai, Y.C., Tél, T., 2011. Transient Chaos: Complex Dynamics in Finite Time Scales, first ed. Springer-Verlag, New York.

Major, P.F., 1978. Predator–prey interactions in two schooling fishes, *Caranx ignobilis* and *Stolephorus purpureus*. Anim. Behav. 26, 760–777.

McCann, K., Yodzis, P., 1995. Bifurcation structure of a three-species food chain model. Theor. Popul. Biol. 48, 93–125.

McCann, K., Yodzis, P., 1994. Nonlinear dynamics and population disappearances. Am. Nat. 144, 873–879.

Mills, M.G.L., 1978. Foraging behaviour of the brown hyaena (*Hyaena brunnea* Thunberg, 1820) in the Southern Kalahari. Z. Tierpsychol. 48, 113–141.

Nakasuji, F., Dyck, V.A., 1984. Evaluation of the role of *Microvelia douglasi atrolineata* (Bergroth) (Heteroptera: Veliidae) as predator of the brown planthopper *Nilaparvata lugens* (Stål) (Homoptera: Delphacidae). Res. Popul. Ecol. 26, 134–149.

Packer, C., Ruttan, L., 1988. The evolution of cooperative hunting. Am. Nat. 132, 159–198.

Rypstra, A.L., 1985. Aggregations of *Nephila clavipes* (L.) (Araneae, Araneidae) in relation to prey availability. J. Arachnol. 13, 71–78.

Sabuco, J., Zambrano, S., Sanjuán, M.A.F., 2010. Partial control of chaotic systems using escape times. New J. Phys. 12, 113038.

Sabuco, J., Sanjuán, M.A.F., Yorke, J.A., 2012a. Dynamics of partial control. Chaos 22, 047507.

Sabuco, J., Zambrano, S., Sanjuán, M.A.F., Yorke, J.A., 2012b. Finding safety in partially controllable chaotic systems. Commun. Nonlinear Sci. Numer. Simulat. 17, 4274–4280.

Schreiber, S.J., 2001. Chaos and sudden extinction in simple ecological models. J. Math. Biol. 42, 239–260.

Sinha, S., Parthasarathy, S., 1996. Unusual dynamics of extinction in a simple ecological model. Proc. Natl. Acad. Sci. U. S. A. 93, 1504–1508.

Stander, P.E., 1991. Cooperative hunting in lions: the role of the individual. Behav. Ecol. Sociobiol. 29, 445–454.

Vandermeer, J., Yodzis, P., 1999. Basin boundary collision as a model of discontinuous change in ecosystems. Ecology 80, 1817–1827.

Ward, P.I., Enders, M.M., 1985. Conflict and cooperation in the group feeding of the social spider *Stegodyphus mimosarum*. Behaviour 94, 167–182.

Weis, J.B., Knobloch, E., 1990. A stochastic return map for stochastic differential equations. J. Stat. Phys. 58, 863–883.

Zambrano, S., Sanjuán, M.A.F., Yorke, J.A., 2008. Partial control of chaotic systems. Phys. Rev. E 77, 055201(R).

Zambrano, S., Sanjuán, M.A.F., 2009. Exploring partial control of chaotic systems. Phys. Rev. E 79, 026217.

Zambrano, S., Sabuco, J., Sanjuán, M.A.F., 2014. How to minimize the control frequency to sustain transient chaos using partial control. Commun. Nonlinear Sci. Numer. Simulat. 19, 726–737.