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Vibrational Resonance in a Duffing System with a Generalized Delayed Feedback

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Abstract

We investigate the vibrational resonance in the Duffing system with different kinds of delayed feedback. Our approach is to consider the delayed feedback as a generalized delayed feedback in a fractional-order differential version. For three special cases, the generalized delayed feedback corresponds to displacement delayed feedback, velocity delayed feedback, and acceleration delayed feedback respectively. At first, based on the vibrational mechanism, the approximate solution of the system is obtained. Then, we give conditions for all resonance patterns. The theoretical predictions are verified by numerical simulations. Furthermore, the theoretical results are in good agreement with the numerical simulations. Since the delayed feedback is in a generalized form, our results can be regarded as universal for the vibrational resonance in nonlinear systems with different kinds of delayed feedback.

1 Introduction

In the last decades, vibrational resonance (VR) has attracted more and more attention. VR was introduced by Landa and McClintock [1]. It is similar to the well-known stochastic resonance phenomenon and focuses on improving the response of a nonlinear system to a weak low-frequency signal by adjusting the other high-frequency signal. Due to the wide applications of biharmonical signals in a variety of disciplines, VR has been thoroughly investigated in a noisy oscillator [2–4],

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neuronal networks [5–7], coupled systems [8–11], delayed systems [12–18], ferroelectric liquid crystal [19], optical lattice [20], ecosystem [21], etc. Recently, VR in fractional system was reported by Yang and Zhu [22], and they found that the fractional-order damping induces new VR patterns. The fractional-order calculus theory is a very useful tool in many scientific and engineering fields [23–28]. Hence, it is important to investigate the dynamical properties of the fractional-order systems further.

Because the time delay feedback and the fractional scheme are both important control strategies, Yang and Zhu [29] investigated VR in the fractional Duffing system with a displacement delayed feedback. However, as far as we know, the delayed feedback has been always considered in previous works on VR as a displacement delayed feedback. The effects of the velocity delayed feedback and the acceleration delayed feedback on VR have never been studied until now. So that, this is the main motivation of this paper. In order to obtain a universal solution, we adopt the concept of generalized fractional-order differential delayed feedback which was proposed by Wang and Zheng [30]. This kind of delayed feedback generalizes the displacement delayed feedback, velocity delayed feedback and acceleration delayed feedback. At first, we introduce the definition of the fractional derivative. There are several definitions for the fractional derivative, but the most popular definitions are the Riemann-Liouville (RL) definition, the Caputo definition, and the Grünwald-Letnikov definition (GL) [31]. For most cases, these three definitions are equivalent. In the following analysis, we use the GL definition for its simplicity in the numerical calculations. The Grünwald-Letnikov definition is given by

$$\mathrm{D}^{\alpha} f(t)|_{t=kh} = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} f(kh-jh), \tag{1}$$

where the binominal coefficients are

$$\binom{\alpha}{0} = 1, \quad \binom{\alpha}{j} = \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!} \quad \text{for} \quad j \geq 1, \tag{2}$$

Here, we study the following Duffing system whose equation is given by

$$\ddot{x}(t) + \delta\dot{x} + \omega_0 x + \beta x^3 + \gamma D^{\alpha} x(t-\tau) = f\cos(\omega t) + F\cos(\Omega t), \tag{3}$$

where δ, ω_0 and β are the system coefficients, $\delta > 0$, $\beta > 0$. The excitations satisfy $f \ll 1$ and $\omega \ll \Omega$. $D^{\alpha}x(t-\tau)$ is the generalized fractional-order differential delayed feedback with fractional-order $\alpha \in [0,2]$, and γ and τ respectively denote the delayed strength and the delay parameter. Here, we assume $\gamma > 0$. For the case α=0, $D^{\alpha}x(t-\tau)$ reduces to the displacement delayed feedback $x(t-\tau)$, and VR in Eq. (3) with this kind of feedback was studied by Jeevarathinam *et al.* [15]. While for the case $\alpha = 1$ and $\alpha = 2$, it turns to the velocity delayed feedback $\dot{x}(t-\tau)$ and the acceleration delayed feedback $\ddot{x}(t-\tau)$ respectively. We will analyze the VR in Eq. (3) with these two kinds of delayed feedback. Except the above special cases, when $\alpha \in (0,1) \cup (1,2)$, it is a generalized differential delayed feedback and usually realized by a circuit in the control loops. In the linear vibration system, the generalized differential delayed feedback can improve the stability of the system [30]. In absence of the external signals, the damping term, and the delayed feedback, the potential function of the Duffing system is $V(x) = \frac{1}{2}(\omega_0+\gamma)x^2 + \frac{1}{4}\beta x^4$ for $\alpha = 0$ and $V(x) = \frac{1}{2}\omega_0 x^2 + \frac{1}{4}\beta x^4$ for $0 < \alpha \leq 2$. If $\alpha = 0$, $V(x)$ has a double-well for the case $\omega_0 + \gamma < 0$ and a single-well for the case $\omega_0 + \gamma > 0$. If $0 < \alpha \leq 2$, $V(x)$ has a double-well for the case $\omega_0 < 0$ and a single-well for the case $\omega_0 > 0$. Hence, the shape of the potential function depends on the fractional-order α. The structure of the paper is organized as follows. At first, a theoretical analysis is carried out to obtain an approximate solution. Then, based on the analytical results, the conditions corresponding to different resonance patterns will be presented. Several examples for the system with different kinds of delayed feedback are investigated by a theoretical analysis and by numerical simulations. Finally, we provide the conclusions.

2 Theoretical formulation

In this section, we give the conditions corresponding to different resonance patterns according to the approximation of the system.

2.1 Approximate solution

In the vibrational mechanism, due to $\omega \ll \Omega$, the method of separation of slow and fast motions can be used to obtain the approximation of the system in Eq. (3) [32]. By this method, we get an approximate but not an accurate solution of the system. Due to its simplicity, this method has been successfully used in systems excited by both low-frequency and high-frequency signals [9, 15, 16, 22, 29, 32–38]. Letting $x = X + \Psi$, where X is the slow motion with period $2\pi/\omega$ and Ψ is the fast motion with period $2\pi/\Omega$, and then substituting it into Eq. (3), one has

$$\begin{aligned}&\ddot{X}+\ddot{\Psi}+\delta\dot{X}+\delta\dot{\Psi}+\omega_0 X+\omega_0\Psi+\beta X^3+3\beta X^2\Psi+3\beta X\Psi^2+\beta\Psi^3+\gamma D^{\alpha}X(t-\tau)+\gamma D^{\alpha}\Psi(t-\tau)\\&=f\cos(\omega t)+F\cos(\Omega t).\end{aligned} \tag{4}$$

Seeking the approximate solution of Ψ in the linear equation

$$\ddot{\Psi}+\delta\dot{\Psi}+\omega_0\Psi+\gamma D^{\alpha}\Psi(t-\tau)=F\cos(\Omega t) \tag{5}$$

and assuming

$$\Psi=\frac{F}{\mu}\Omega^{\alpha}\cos(\Omega t+\phi), \tag{6}$$

one easily has

$$\mu^2=[\omega_0-\Omega^2+\gamma\Omega^{\alpha}\cos(\frac{\alpha\pi}{2}-\Omega\tau)]^2+[\delta\Omega+\gamma\Omega^{\alpha}\sin(\frac{\alpha\pi}{2}-\Omega\tau)]^2, \tag{7}$$

and

$$\phi=-\tan^{-1}\frac{\delta\Omega+\gamma\Omega^{\alpha}\sin(\frac{\alpha\pi}{2}-\Omega\tau)}{\omega_0-\Omega^2+\gamma\Omega^{\alpha}\cos(\frac{\alpha\pi}{2}-\Omega\tau)}. \tag{8}$$

Substituting the solution of Ψ into Eq. (4) and averaging all terms over the interval $[0, 2\pi/\Omega]$, one gets the equation for the slow motion

$$\ddot{X}+\delta\dot{X}+C_1X+\beta X^3+\gamma D^{\alpha}X(t-\tau)=f\cos(\omega t), \tag{9}$$

where $C_1 = \omega_0 + \frac{3\beta F^2}{2\mu^2}$.

(i) The double-well potential case.

When $f=0$, there is a critical point F_c which transforms Eq. (9) from bistable to monostable. For the case $\alpha=0$, the critical point is $F_c=\sqrt{-\frac{2\mu^2(\omega_0+\gamma)}{3\beta}}$. When $F<F_c$, the stable equilibrium points of Eq. (9) are $X_{\pm}^*=\pm\sqrt{-\frac{C_1+\gamma}{\beta}}$. When $F\geq F_c$, the stable equilibrium point of Eq. (9) is $X_0^*=0$. While for the case $0<\alpha\leq 2$, the critical point is $F_c=\sqrt{-\frac{2\mu^2\omega_0}{3\beta}}$ and the stable equilibrium points are $X_{\pm}^*=\pm\sqrt{-\frac{C_1}{\beta}}$ for $F<F_c$ and $X_0^*=0$ for $F\geq F_c$.

(ii) The single-well potential case.

Since $C_1>0$, $\beta>0$ and $\gamma>0$, there is only one equilibrium point $X_0^*=0$.

The slow oscillation occurs around the stable equilibrium point. Considering the deviation Y of X from X^*, i.e., $Y=X-X^*$, one gets

$$\ddot{Y}+\delta\dot{Y}+\omega_r^2 Y+3\beta X^*Y^2+\beta Y^3+\gamma D^\alpha Y(t-\tau)=f\cos(\omega t), \tag{10}$$

where $\omega_r^2=C_1+3\beta X^{*2}$. For a weak signal, $f\ll 1$, one ignores all nonlinear terms in Eq. (10) and obtains the linear equation

$$\ddot{Y}+\delta\dot{Y}+\omega_r^2 Y+3\beta X^*Y^2+\beta Y^3+\gamma D^\alpha Y(t-\tau)=f\cos(\omega t). \tag{11}$$

Solving Eq. (11), one obtains the solution of X in the limit $t\to\infty$ as $Y=A_L cos(\omega t+\theta)$, where

$$\begin{aligned}
&A_L=\frac{f}{\sqrt{S}},\\
&S=[\omega_r^2-\omega^2+\gamma\omega^\alpha\cos(\frac{\alpha\pi}{2}-\omega\tau)]^2+[\delta\omega+\gamma\omega^\alpha\sin(\frac{\alpha\pi}{2}-\omega\tau)]^2,\\
&\theta=-\tan^{-1}\frac{\delta\omega+\gamma\omega^\alpha\sin(\frac{\alpha\pi}{2}-\omega\tau)}{\omega_r^2-\omega^2+\gamma\omega^\alpha\cos(\frac{\alpha\pi}{2}-\omega\tau)}.
\end{aligned} \tag{12}$$

To quantify the VR effect, the response amplitude Q is often considered as a measurement, which is defined by $Q=A_L/f$. According to Eq. (12), the response amplitude Q for the system with generalized fractional-order differential delayed feedback is given by

$$Q=\frac{1}{\sqrt{[\omega_r^2-\omega^2+\gamma\omega^\alpha\cos(\frac{\alpha\pi}{2}-\omega\tau)]^2+[\delta\omega+\gamma\omega^\alpha\sin(\frac{\alpha\pi}{2}-\omega\tau)]^2}}. \tag{13}$$

We obtain the response amplitude Q of the system with the displacement delayed feedback, velocity delayed feedback and acceleration delayed feedback respectively, by substituting $\alpha=0,1$ and 2 into Eq. (13).

2.2 Resonance analysis

From the definition of Q, we know that the resonance occurs when the denominator in Eq. (13) achieves a minimal value. In the following analysis, we use F as a control parameter. In Eq. (13), F is only included in the term ω_r^2. Letting $S_1=\left[\omega_r^2-\omega^2+\gamma\omega^\alpha\cos\left(\frac{\alpha\pi}{2}-\omega\tau\right)\right]^2$, Q has a maximal value when S_1 has a minimal value and as a result the resonance occurs. According to Eq. (13), we obtain the following results:

(i) The double-well potential case.
Since the stable equilibrium point X^*, which depends on α, is included in ω_r^2 in Eq. (13), and we have the two following cases.
(a) The case $\alpha = 0$.
When

$$0 < \omega^2 < \gamma \cos \omega\tau - 3\gamma - 2\omega_0, \tag{14}$$

the resonance occurs at the $F_{VR}^{(1)}$ and $F_{VR}^{(2)}$ which are two real roots of $S_1 = 0$, i.e.,

$$F_{VR}^{(1)} = \sqrt{\frac{\mu^2(\gamma \cos \omega\tau - \omega^2 - 3\gamma - 2\omega_0)}{3\beta}} < F_c, \tag{15}$$

and

$$F_{VR}^{(2)} = \sqrt{\frac{2\mu^2(\omega^2 - \gamma \cos \omega\tau - \omega_0)}{3\beta}} > F_c. \tag{16}$$

The peak values of Q at these two points are identical, i.e.,

$$Q_{\max}^{(1)} = Q_{\max}^{(2)} = \frac{1}{|\gamma \sin(\omega\tau) + \delta\omega|}. \tag{17}$$

When

$$\omega^2 \geq \gamma \cos \omega\tau - 3\gamma - 2\omega_0 \tag{18}$$

the resonance occurs only at $F_{VR}^{(2)}$. The peak value of Q is also expressed in Eq. (17). Here, the results correspond to the resonance pattern in the system with displacement delayed feedback.

(b) The case $0 < \alpha \leq 2$.

When

$$0 < \omega^2 \leq \gamma\omega^\alpha \cos(\frac{\alpha\pi}{2} - \omega\tau), \tag{19}$$

$S_1 = 0$ has no real roots, and the resonance occurs at $F_{VR} = F_c = \sqrt{-\frac{2\mu^2\omega_0}{3\beta}}$. The peak value of Q is

$$Q_{\max} = \frac{1}{\sqrt{\left[\gamma\omega^\alpha \cos\left(\frac{\alpha\pi}{2} - \omega\tau\right) - \omega^2\right]^2 + \left[\gamma\omega^\alpha \sin\left(\frac{\alpha\pi}{2} - \omega\tau\right) + \delta\omega\right]^2}}. \tag{20}$$

When

$$\gamma\omega^\alpha \cos(\frac{\alpha\pi}{2} - \omega\tau) < \omega^2 < \gamma\omega^\alpha \cos(\frac{\alpha\pi}{2} - \omega\tau) - 2\omega_0, \tag{21}$$

the resonance occurs at $F_{VR}^{(1)}$ and $F_{VR}^{(2)}$ which are two real roots of $S_1 = 0$, i.e.,

$$F_{VR}^{(1)} = \sqrt{\frac{\mu^2\left[\gamma\omega^\alpha \cos\left(\frac{\alpha\pi}{2} - \omega\tau\right) - 2\omega_0 - \omega^2\right]}{3\beta}} < F_c, \tag{22}$$

and

$$F_{VR}^{(2)} = \sqrt{\frac{2\mu^2\left[\omega^2 - \omega_0 - \gamma\omega^\alpha \cos\left(\frac{\alpha\pi}{2} - \omega\tau\right)\right]}{3\beta}} > F_c. \tag{23}$$

The peak values of Q at these two points are identical, i.e.,

$$Q_{\max}^{(1)} = Q_{\max}^{(2)} = \frac{1}{\left|\delta\omega + \gamma\omega^{\alpha}\sin(\frac{\alpha\pi}{2} - \omega\tau)\right|}. \tag{24}$$

When

$$\omega^2 \geq \gamma\omega^{\alpha}\cos(\frac{\alpha\pi}{2} - \omega\tau) - 2\omega_0, \tag{25}$$

The resonance occurs at $F_{VR}^{(2)}$. The peak value of Q is also expressed in Eq. (24). For the special case $\alpha = 1$ and $\alpha = 2$, the results in this subsection correspond to the resonance pattern in the system with velocity and acceleration delayed feedback respectively.
(ii) The single-well potential case.
For this case, the slow motion occurs around the stable equilibrium point $X^* = 0$. When

$$\omega^2 > \omega_0 - \gamma\omega^{\alpha}\cos(\frac{\alpha\pi}{2} - \omega\tau), \tag{26}$$

the resonance occurs at

$$F_{VR} = \sqrt{\frac{2\mu^2[\omega^2 - \omega_0 - \gamma\omega^{\alpha}\cos(\frac{\alpha\pi}{2} - \omega\tau)]}{3\beta}}. \tag{27}$$

The peak value of the response amplitude Q is also expressed by Eq. (24). Elsewhere, there is no resonance, and the response amplitude decreases with the value of F. At $F = 0$, the response amplitude Q achieves the maximal value

$$Q_{\max} = \frac{1}{\sqrt{[\omega_0 - \omega^2 + \gamma\omega^{\alpha}\cos(\frac{\alpha\pi}{2} - \omega\tau)]^2 + [\delta\omega + \gamma\omega^{\alpha}\sin(\frac{\alpha\pi}{2} - \omega\tau)]^2}}. \tag{28}$$

The results in Eqs. (26)–(28) holds true for all values of $\alpha \in [0,2]$. In other words, for the single-well potential case, the results are suitable for the system with displacement delayed feedback, velocity delayed feedback, acceleration delayed feedback and generalized fractional-order differential delayed feedback.

3 Numerical simulations

In this section, we give some numerical examples to verify the theoretical predictions shown in Sec. 2. For the numerical simulations, the response amplitude Q is computed by using the formula

$$Q = \sqrt{Q_{\sin}^2 + Q_{\cos}^2}/f, \tag{29}$$

where $Q_{\sin} = \frac{2}{rT}\int_0^{rT} x(t)\sin(\omega t)dt$ and $Q_{\cos} = \frac{2}{rT}\int_0^{rT} x(t)\cos(\omega t)dt$. The period $T = 2\pi/\omega$ and r is a positive integer which should be chosen big enough. To obtain the time series $x(t)$, we use the GL definition to discretize Eq. (3).

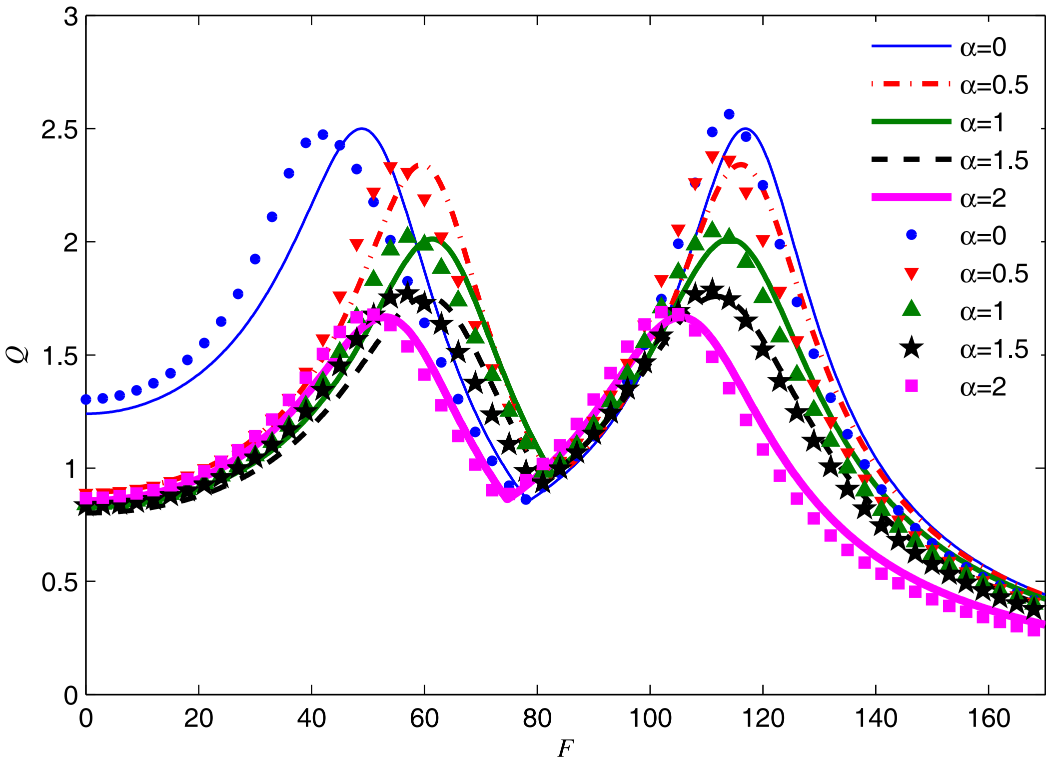

Fig. 1 The double-resonance occurs at $F_{VR}^{(1)}$ and $F_{VR}^{(2)}$ for $\delta = 0.5$, $\omega_0 = -1$, $\beta = 1$, $\gamma = 0.1$, $\tau = 1.6$, $f = 0.1$, $\omega = 1$ and $\Omega = 10$. The lines are the analytical results while the markers are the numerical results.

3.1 The double-well potential case

In Fig. 1, the parameters satisfy the condition shown in Eq. (14) for $\alpha = 0$ and in Eq. (21) for $0 < \alpha \leq 2$. It leads to a double-resonance at $F_{VR}^{(1)}$ and $F_{VR}^{(2)}$. The peak values of Q are given in Eq. (17) and Eq. (24) respectively for $\alpha = 0$ and $0 < \alpha \leq 2$. The variation of the fractional-order α, i.e., the kind of the delayed feedback, cannot induce changes in the resonance pattern. The value of the fractional-order α mainly influences the position of F_{VR} and the value of $Q_{\max}$. In this figure, the analytical results are in good agreement with the numerically calculated ones, especially when $\alpha > 0$. The errors between the two kinds of results induced by the fractional-order α will be explained later.

In Fig. 2, a single-resonance pattern is clearly shown. In this figure, a single-resonance occurs at $F_{VR}^{(2)}$ and the response amplitude arrives at a minimal at F_c. Compared with the simulation parameters in Fig. 1, we only change the parameter ω_0 from -1 to -0.2. Hence, the linear stiffness ω_0 has an important effect on the resonance pattern. The change of ω_0 can make the resonance pattern to transform from a double-resonance to a single-resonance pattern. This is due to the fact that the conditions in Eq. (18) and Eq. (25) are respectively satisfied for $\alpha = 0$ and $0 < \alpha \leq 1$ when $\omega_0 = -0.2$. The response amplitude Q reaches the minimal value at the critical point F_c. The resonance pattern is also independent of the fractional-order α for chosen parameters in Fig. 2.

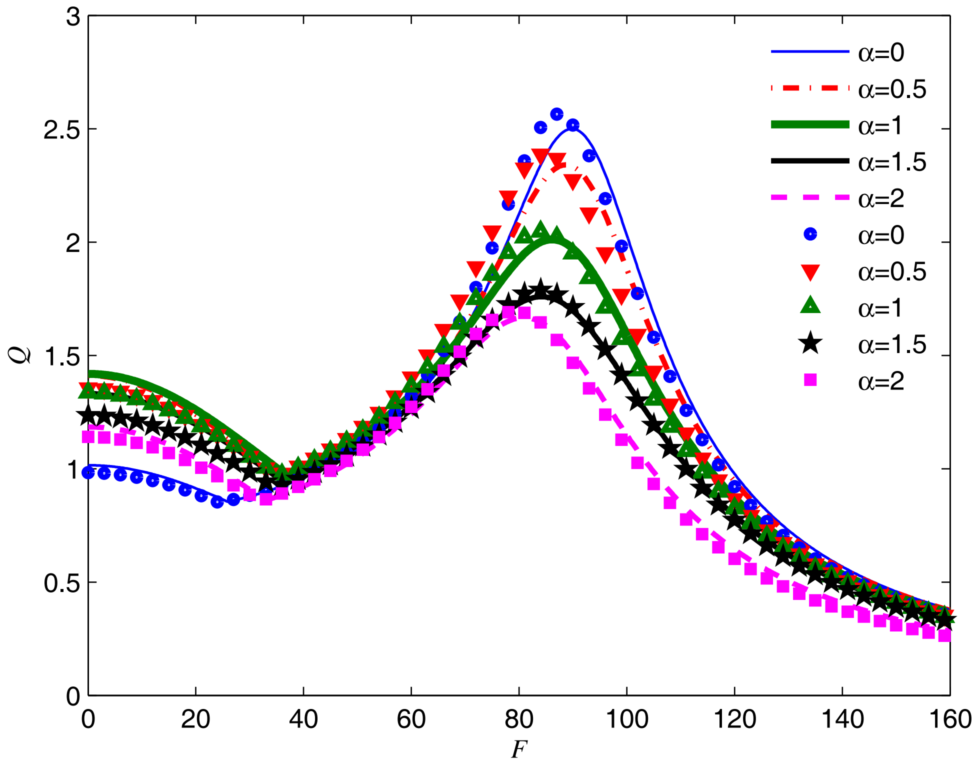

Fig. 2 The single-resonance occurs at $F_{VR}^{(2)}$ for $\delta = 0.5$, $\omega_0 = -0.2$, $\beta = 1$, $\gamma = 0.1$, $\tau = 1.6$, $f = 0.1$, $\omega = 1$ and $\Omega = 10$. The lines are the analytical results while the markers are the numerical results.

3.2 The single-well potential case

In Fig. 3, a single-resonance pattern in the system with a single-well potential is shown. The parameters in this figure satisfy the condition shown in Eq. (26). It results that a single-resonance occurs at F_{VR} which is given in Eq. (27). The fractional-order α mainly affects the peak value of Q in this figure. The peak value Q_{max} decreases with the increase of α. In other words, the displacement delayed feedback can improve the weak low-frequency signal better than the acceleration delayed feedback.

In Fig. 4, a no-resonance pattern is shown. The parameters chosen in this figure do not satisfy the condition shown in Eq. (26). It results that the response amplitude Q decreases with the increase of the value of F. There is no resonance no matter we choose the displacement delayed feedback, the velocity delayed feedback, the acceleration delayed feedback or the generalized fractional-order differential delayed feedback. With the increase of F, the response amplitude Q decreases to a small value.

The results in this section show that the theoretical predictions are in good agreement with the numerical simulations, and the error between the two results are very small, proving the correctness of our analysis. The small error is induced by several factors. The first factor is the vibrational mechanism used in this paper. The method of separation of slow and fast motions is an approximate approach. In order to obtain the solution in a simple way, the slow motion is considered as a constant

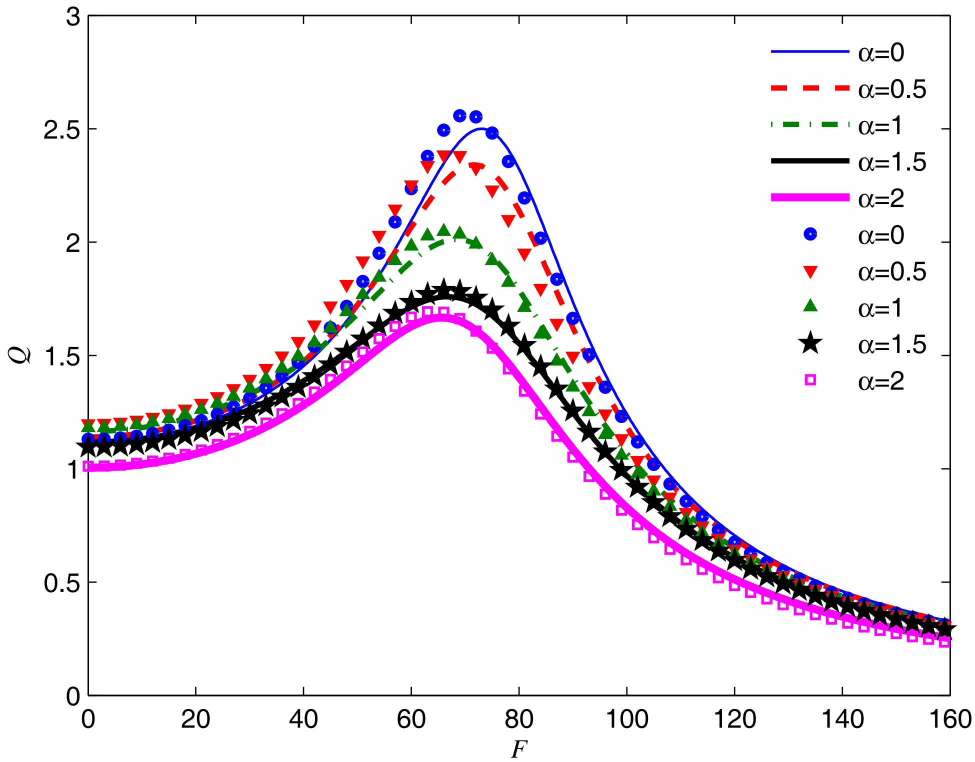

Fig. 3 The single-resonance occurs at F_{VR} for $\delta = 0.5$, $\omega_0 = 0.2$, $\beta = 1$, $\gamma = 0.1$, $\tau = 1.6$, $f = 0.1$, $\omega = 1$ and $\Omega = 10$. The lines are the analytical results while the markers are the numerical results.

in the period of the fast motion and some nonlinear terms are ignored in the solving procedure. Moreover, the system is sensitive to the external perturbations for the parametric values near the transition from bistability to monostability. The error brought by these approximations has been discussed in the previous references [15, 32, 38]. The second factor able to induce errors is the potential function of the system. For the theoretical analysis, the potential of the system is $V(x) = \frac{1}{2}(\omega_0 + \gamma)x^2 + \frac{1}{4}\beta x^4$ when $\alpha = 0$ and $V(x) = \frac{1}{2}\omega_0 x^2 + \frac{1}{4}\beta x^4$ when $0 < \alpha \leq 2$, since we consider the fractional-order derivative as a damping term when $0 < \alpha < 2$ while a state variable term when $\alpha = 0$. In fact, the fractional-order derivative is a term that behaves as both a damping and a state variable when $0 < \alpha < 1$. Hence, the potential function should contain the fractional-order α when $0 < \alpha < 1$. The potential function varies gradually with the fractional-order in the interval $\alpha \in (0, 1)$. The term α disappears in the potential function for $\alpha = 0$ and $1 \leq \alpha \leq 2$. Hence, the potential function considered as $V(x) = \frac{1}{2}\omega_0 x^2 + \frac{1}{4}\beta x^4$ is inaccurate for $\alpha \in (0, 1)$. The third factor inducing errors comes from the numerically computed process. Different algorithms, computation times, time steps, etc, can also bring errors. Even though many factors might contribute to errors, ignoring them is not a problem. Our verification between the theoretical and numerical results have shown it.

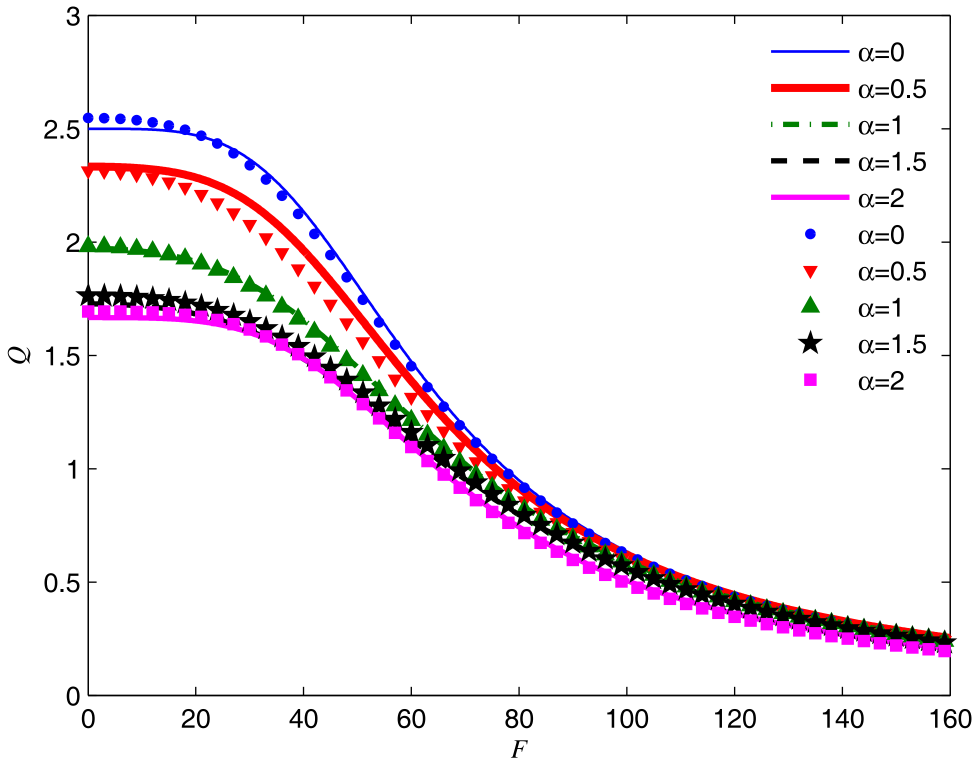

Fig. 4 There is no resonance for $\delta = 0.5$, $\omega_0 = 1$, $\beta = 1$, $\gamma = 0.1$, $\tau = 1.6$, $f = 0.1$, $\omega = 1$ and $\Omega = 10$. The lines are the analytical results while the markers are the numerical results.

4 Conclusions

In this paper, the vibrational resonance is investigated in a Duffing system with different kinds of delayed feedback. In order to obtain a general solution, we considered the delayed feedback as a generalized differential delayed feedback. The displacement delayed feedback, the velocity delayed feedback and the acceleration delayed feedback are special cases of our generalized differential delayed feedback. Moreover, except these special cases, the generalized differential delayed feedback is also a control strategy in the field of engineering. At first, based on the vibrational mechanism, we obtain the approximate general solution of the system for different cases of the fractional-order by the method of separation of slow and fast motions. Then, the conditions for different resonance patterns are given. For the double-well potential case, the resonance patterns contain a double-resonance and two different kinds of single-resonance. For the single-well potential case, the resonance patterns contain a single-resonance and a no-resonance pattern. The value of the fractional-order mainly influences the location and the magnitude of the resonance peak. Finally, our theoretical predictions have been verified by the numerical simulations. The results obtained by the two methods are in good agreement, what proves the correctness of our analysis. Since the systems with displacement delayed feedback, velocity delayed feedback and acceleration delayed feedback are widely used in many fields, we believe that our results are of a general nature for vibrational resonance in delayed systems.

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