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TO ESCAPE OR NOT TO ESCAPE, THAT IS THE QUESTION — PERTURBING THE HÉNON–HEILES HAMILTONIAN

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In this work, we study the Hénon–Heiles Hamiltonian, as a paradigm of open Hamiltonian systems, in the presence of different kinds of perturbations as dissipation, noise and periodic forcing, which are very typical in different physical situations. We focus our work on both the effects of these perturbations on the escaping dynamics and on the basins associated to the phase space and to the physical space. We have also found, in presence of a periodic forcing, an exponential-like decay law for the survival probability of the particles in the scattering region where the frequency of the forcing plays a crucial role. In the bounded regions, the use of the OFLI2 chaos indicator has allowed us to characterize the orbits. We have compared these results with the previous ones obtained for the dissipative and noisy case. Finally, we expect this work to be useful for a better understanding of the escapes in open Hamiltonian systems in the presence of different kinds of perturbations.

Keywords: Nonlinear dynamics and chaos; fractals; numerical simulation of chaotic systems.

1. Introduction

The Hénon–Heiles system [Hénon & Heiles, 1964] represents a paradigmatic model for time-independent Hamiltonian systems with two degrees of freedom [Barrio *et al.*, 2008, 2009]. This system defines the motion of a particle with unit mass in the two-dimensional potential $V(x) = \frac{1}{2}(x^2 + y^2) + x^2y - \frac{1}{3}y^3$ and its corresponding Hamiltonian reads

$$\mathcal{H}_0 = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(x^2 + y^2) + x^2y - \frac{1}{3}y^3. \quad (1)$$

Two main types of motion exist for different values of the energy: bounded and unbounded motion. There is a threshold value of the energy E ($\mathcal{H}_0$ for the unperturbed case), called *escape energy*, $E_e = 1/6$, for which the particle might escape from the potential well for values of energy above it. The system presents a triangular symmetry D_3 with three different exits for which the particle may escape (see Fig. 1 and [Hénon & Heiles, 1964; Barrio *et al.*, 2009; Aguirre *et al.*, 2001; Seoane *et al.*, 2006]).

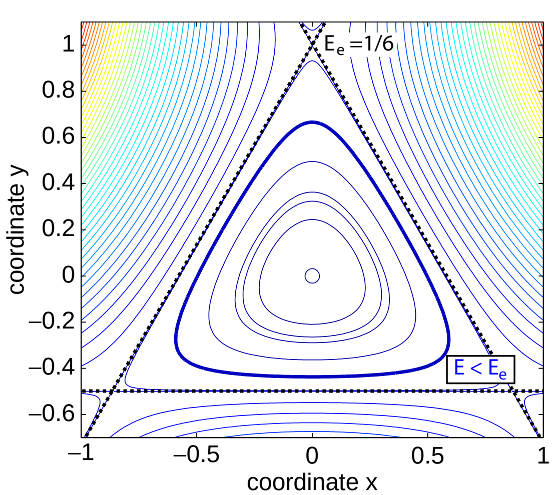
The equations of motion read

$$\begin{aligned} \ddot{x} + x + 2xy &= 0, \\ \ddot{y} + y + x^2 - y^2 &= 0. \end{aligned} \tag{2}$$

A basic property of the Hénon–Heiles system is the existence of a class of highly unstable periodic orbits for $E > E_e$, called the *Lyapunov orbits* [Contopoulos, 1990], that live near the border of the scattering region [Fig. 1(b)] which is the region where the particle interacts with the potential. When a trajectory crosses one of these periodic orbits from inside, it scatters off to infinity and never comes back. The Lyapunov orbits thus provide a meaningful criterion for measuring the delay times of particles, namely escapes times T, in the scattering region even when the system is dissipative [Contopoulos, 1990]. Related to this, and as a well known result for the conservative case, is the algebraic decay law [Karney, 1983] for the survival probability of the particles in the scattering region.

On the other hand, the phase space in the Hénon–Heiles system [Barrio *et al.*, 2008; Aguirre *et al.*, 2001] has a very rich fractal structure. Our system also has a strong topological property, the Wada property. This topological property has been shown in several dynamical systems [Poon *et al.*, 1996; Kennedy & Yorke, 1991; Aguirre *et al.*, 2009], and it is typical in open hamiltonian systems with three or more exits. To understand what Wada basins are, we introduce the concepts of both basins of attraction and exit basin. A basin of attraction is the set of initial conditions that leads to an attractor while an exit basin is the set of initial conditions that lead to a certain exit. After this and from a mathematical point of view, a basin is Wada if any boundary point also belongs to the boundary of at least two other basins [Kennedy & Yorke, 1991]. This phenomenon has been observed in numerical simulations of the Hénon–Heiles system in the phase space [Aguirre *et al.*, 2001] and also in the physical space [Barrio *et al.*, 2008].

The effect of weak perturbations such as noise and dissipation have been studied previously in some specific physical contexts [Seoane *et al.*, 2006; Seoane *et al.*, 2007; Seoane & Sanjuán, 2008, 2010; Seoane *et al.*, 2009] but the effect of a periodic forcing on this kind of systems has not been the focus of study in the last years. Since this kind of perturbation may have implications in problems concerning internal oscillations or the effect of companion galaxies [Kandrup & Novotny, 2004], among others, we have studied its effect in this paradigmatic



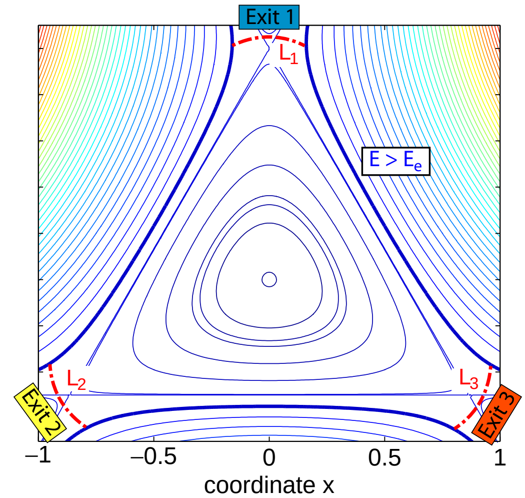


Fig. 1. Level curves of the Hénon–Heiles potential, exits, limit of bounded motion and the three Lyapunov orbits L_i on the (x, y) plane for (a) $E < E_e$ and (b) $E > E_e$.

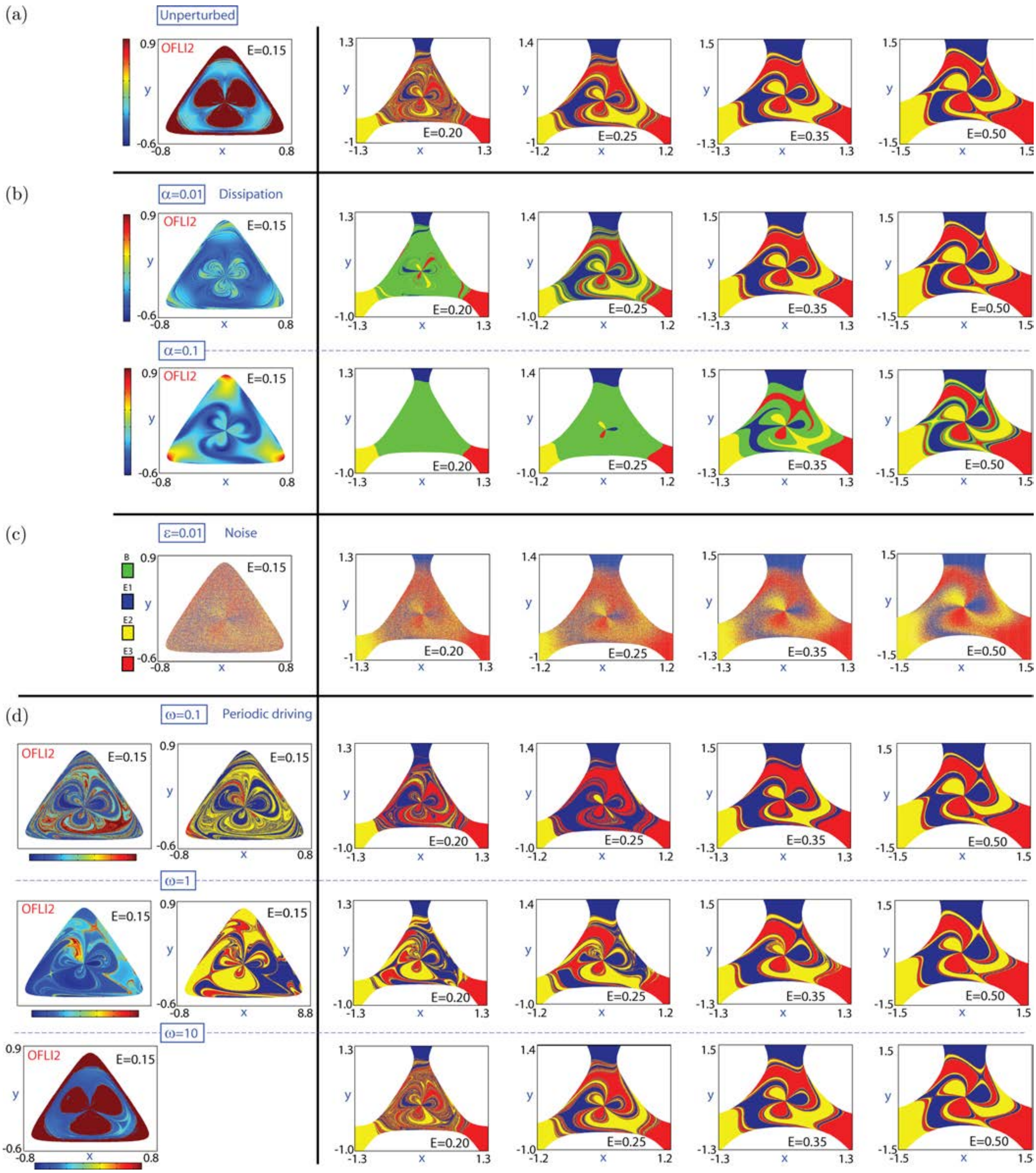


Fig. 2. Plot of the exit basins and the basins of attraction of the Hénon–Heiles Hamiltonian in the physical space (x, y) for the energy values indicated in every figure. The color code is as follows: B corresponds to bounded orbits, and E_1, E_2 and E_3 correspond to the initial condition that leads to exit numbers 1, 2 and 3, respectively. (a) Corresponds to the unperturbed Hamiltonian. In (b) the values of the dissipative parameter are $\alpha = 0.01$ (top) and $\alpha = 0.1$ (bottom). (c) Is the noisy problem with noise intensity $\varepsilon = 0.01$. And finally, in (d) forcing amplitude $A = 0.1$ and frequency values (i) $\omega = 0.1$, (ii) $\omega = 1$ and $\omega = 10$, respectively. Notice that the exit basins in the periodic driving case, for $E = 0.15$ and $\omega = 10$, have not been plotted since for those parameter values the system does not present escapes.

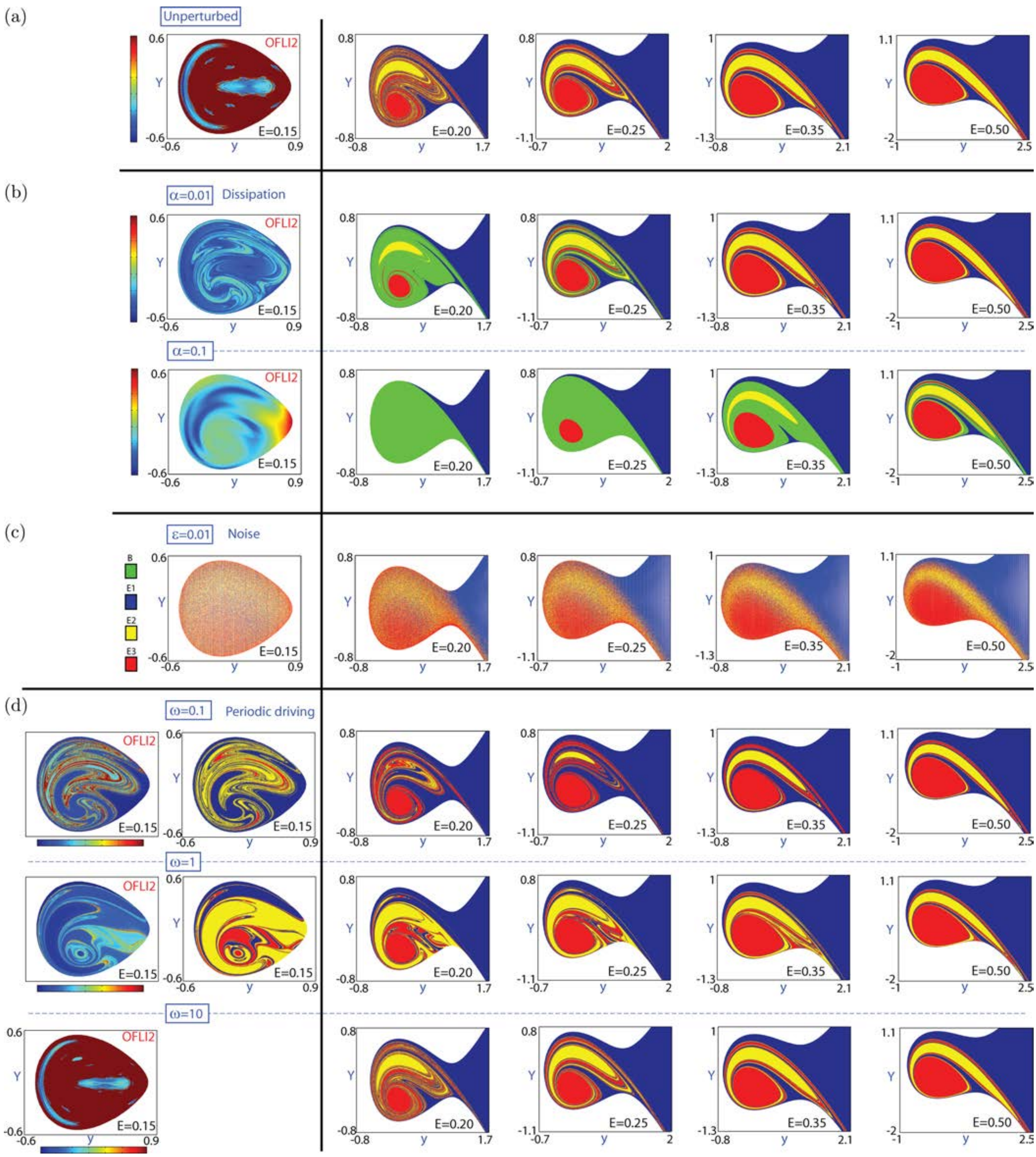


Fig. 3. Plot of the exit basins and the basins of attraction of the Hénon–Heiles Hamiltonian in the phase space (y, Y) for the energy values indicated in every figure. The color code is as follows: B corresponds to bounded orbits, and E_1, E_2 and E_3 correspond to the initial condition that leads to exit numbers 1, 2 and 3, respectively. Notice that $Y = \dot{y}$. (a) Corresponds to the unperturbed Hamiltonian. In (b) the values of the dissipative parameter are $\alpha = 0.01$ (top) and $\alpha = 0.1$ (bottom). (c) Is the noisy problem with noise intensity $\varepsilon = 0.01$. And finally, in (d) forcing amplitude $A = 0.1$ and frequency values (i) $\omega = 0.1$, (ii) $\omega = 1$ and $\omega = 10$, respectively. Notice that, as in Fig. 2, the exit basins in the periodic driving case, for $E = 0.15$ and $\omega = 10$, have not been plotted since for those parameter values the system does not present escapes.

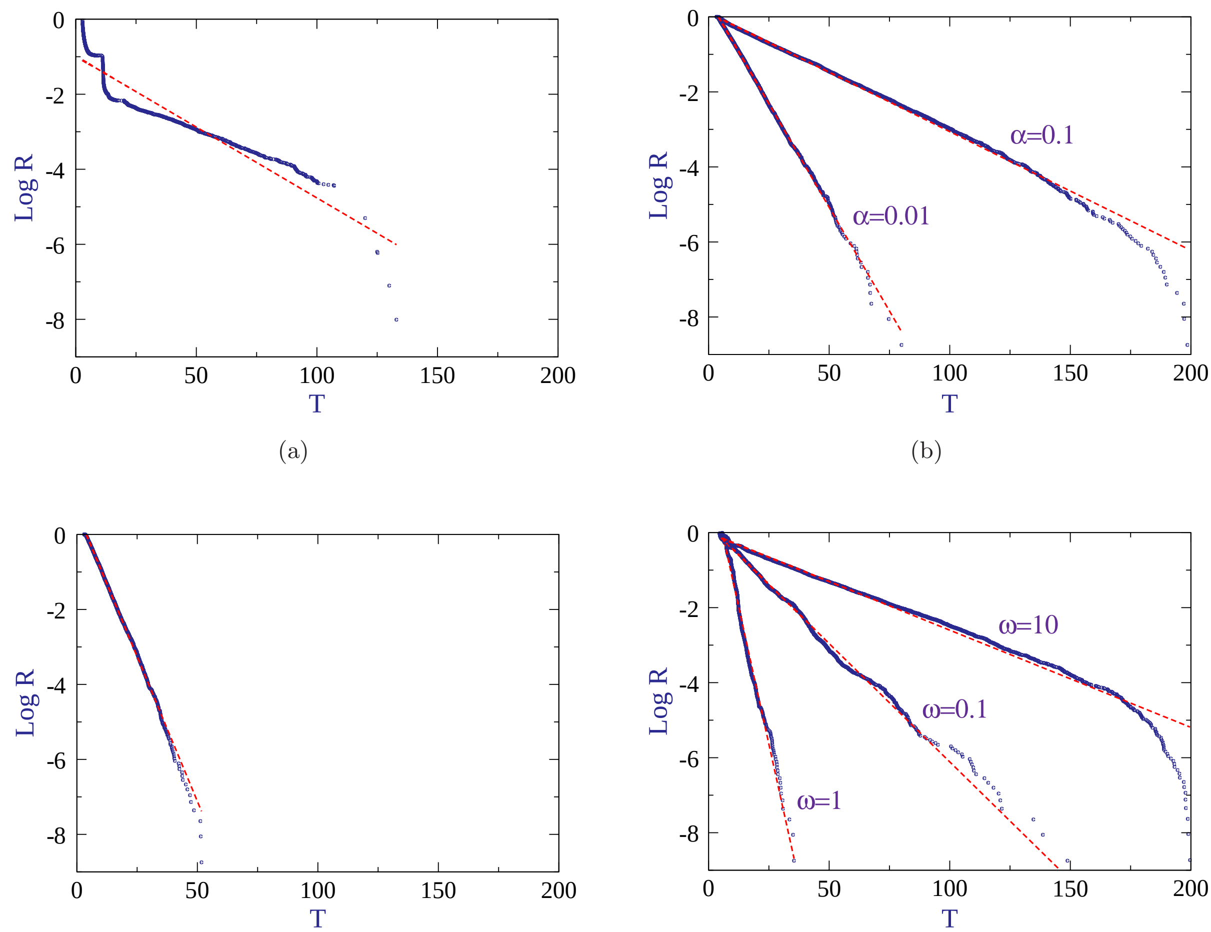
model. In this work, we have also added the effect of both noise and dissipation on the basin structure in the physical (x, y) and phase $(y, \dot{y})$ space for a better understanding of the role of these perturbations in systems with escapes.

In the presence of all these perturbations, namely noise, dissipation and forcing, the equations of motion can be written as follows:

$$\ddot{x} = -\frac{\partial H_0}{\partial x} - \alpha_x \dot{x} + A_x \sin(\omega_x t) + \sqrt{2\varepsilon}\xi(t),$$
$$\ddot{y} = -\frac{\partial H_0}{\partial y} - \alpha_y \dot{y} + A_y \sin(\omega_y t) + \sqrt{2\varepsilon}\eta(t), \tag{3}$$

where α_x and α_y are the dissipative parameters, A_x and A_y the amplitude of the driving, ω_x and ω_y the frequency of the driving, ε the intensity of the noise and $\xi(t)$ and $\eta(t)$ random variables. For our numerical simulations and without any loss of generality in the results obtained we have taken $\alpha_x = \alpha_y = \alpha$, $A_x = A_y = A$ and $\omega_x = \omega_y = \omega$. Note that if $\alpha_x \neq \alpha_y$ the D_3 symmetry is not conserved, whereas the other perturbed cases break immediately the symmetry also for $A_x = A_y = A$ and $\omega_x = \omega_y = \omega$ (but the results will differ faster from the symmetric case).

Figures 2 and 3 show the exit basins and basins of attraction in the physical and the phase space using the boundary limits of the unperturbed case (the zero velocity curves). Each block — (a), (b), (c) and (d) — corresponds to the unperturbed case, with dissipation, with noise or with a periodic



(a) (b) (c) (d)

Fig. 4. Typical exponential decay law for the particles remaining in the scattering region. R denotes the fraction of particles remaining in the scattering region. Here, we shoot 5×10^3 with energy $E = 0.2$ from $(x_0, y_0) = (0, -0.5)$ and $\theta \in (0, 2\pi)$. (a) Algebraic law of the unperturbed system. (b) In presence of dissipation. The dissipative parameter is $\alpha = 0.01$ or $\alpha = 0.1$. (c) Due to the noise effects. The intensity of the noise $\varepsilon = 0.01$ and (d) with a periodic driving. The forcing amplitude is $A = 0.1$ and the forcing frequency is $\omega = 1$ (resonant case), $\omega = 0.1$ or (c) $\omega = 10$. The oscillations around the straight line obtained from the linear regression of the numerical data is due to the value of the chosen frequency ω.

forcing, respectively. In the first column, that corresponds to a value of the energy below the escape energy of the unperturbed case, we plot the OFLI2 chaos indicator [Barrio, 2006; Barrio *et al.*, 2010] if the orbits are still bounded, i.e. if the perturbation does not cause the escape basins to appear. The color code of the chaos indicator is the following: red corresponds to chaotic orbits and blue to stable orbits. Note that when OFLI2 is used, it is indicated on the picture, all the other cases are the exit basins and basins of attraction. In Fig. 2 the initial conditions are taken to maintain the symmetry of the unperturbed case, while in Fig. 3 the initial conditions are $(x, y, \dot{x}, \dot{y}) = (0, y, \dot{x}(E, y, \dot{y}), \dot{y})$ (obtaining $\dot{x}$ from the energy E and $y, \dot{y}$).

Figure 4 shows the decay law for the particles remaining in the scattering region. Each plot corresponds to the same perturbations as the blocks: (a), (b), (c) and (d) of previous Figs. 2 and 3. For simulation convenience, we launch scattering particles from within the scattering region and examine their escaping trajectories. Specifically, the particles are distributed on a vertical line segment centered at $(x, y) = (0, 0)$, and they start their motions in different directions. That is, the subspace in the phase space from which scattering particles are initiated can be denoted by (y, θ), where θ is the angle of the initial velocity with respect to the x axis. In all the decay law pictures we shoot 5×10^3 orbits with $\theta \in (0, 2\pi)$.

We analyze separately the effects of every perturbation indicating our contribution in the present field in relation with the results obtained previously by others. For this purpose, this paper is organized as follows. In Sec. 2 we describe our system in the absence of perturbations. Section 3 presents the effects of the dissipation in both the scattering dynamics and the basin topology. The influence of noise on the basin structure and on the escaping dynamics is shown in Sec. 4. On the other hand, the periodic driving case is discussed in Sec. 5. Finally, conclusions and a discussion of the main results of the paper are presented in Sec. 6.

2. Unperturbed Case

First of all, we consider the Hamiltonian case without perturbations. Figure 2(a) shows, for energy values of $E = 0.15$, $E = 0.20$, $E = 0.25$, $E = 0.35$ and $E = 0.50$ the corresponding basins in the physical space and in Fig. 3(a) in the phase space. The color code denotes the set of initial conditions that are escaping through one of the exits of our system and the set of bounded orbits.

At the top of Figs. 2 and 3 (plot a) we show the plates of the unperturbed case in which Wada basins are clearly observed. For energy above $E \approx 0.22$ all the KAM regime disappears [Barrio *et al.*, 2008] and the orbits escape through one of the exits. For energy $E = 0.15$, as all the orbits are bounded, the chaos indicator shows where the regular orbits are. Notice that, in Figs. 2(a) and 3(a), the exit basins become smoother and its fractality decreases when the energy E increases, in the sense that the fat-fractal exponent increases, approaching the value of 1 — no fractal geometry — insofar the energy of the system is large enough (see for more details [Barrio *et al.*, 2008]). This decrease in the fractality has consequences with the unpredictability in the evolution of the system. Here it means that, for high values of the energy, the particles escape very fast from the scattering region and, consequently, the system predictability increases.

In Fig. 4(a), we show the scaling law that corresponds to an algebraic law as expected for the conservative system [Karney, 1983].

3. Dissipative Case

The effect of dissipation in systems with escapes is relevant to physical situations such as the advection of inertial particles in open chaotic flows [Babiano *et al.*, 2004]. Previous work [Motter & Lai, 2002] used a dissipative two-dimensional model to show that small dissipation produces an abrupt transition in the decay law of the particles in the scattering region from an algebraic law to an exponential law. In [Seoane *et al.*, 2007] the authors extend this result for continuous-time models. This exponential behavior can be observed in Fig. 4(b). This result is due to the fact that the dissipation destroys the KAM islands converting them into attractors [Motter & Lai, 2002] and as a consequence the algebraic law into an exponential law.

Other important physical consequences of the dissipation effect is the persistence of the Wada basins in phase space when the dissipative parameter is small enough [Seoane *et al.*, 2006]. Once the dissipation is large enough, the Wada basins are destroyed and the system loses its unpredictability as most of the particles fall into the attractor. In this system, the origin is an equilibrium point

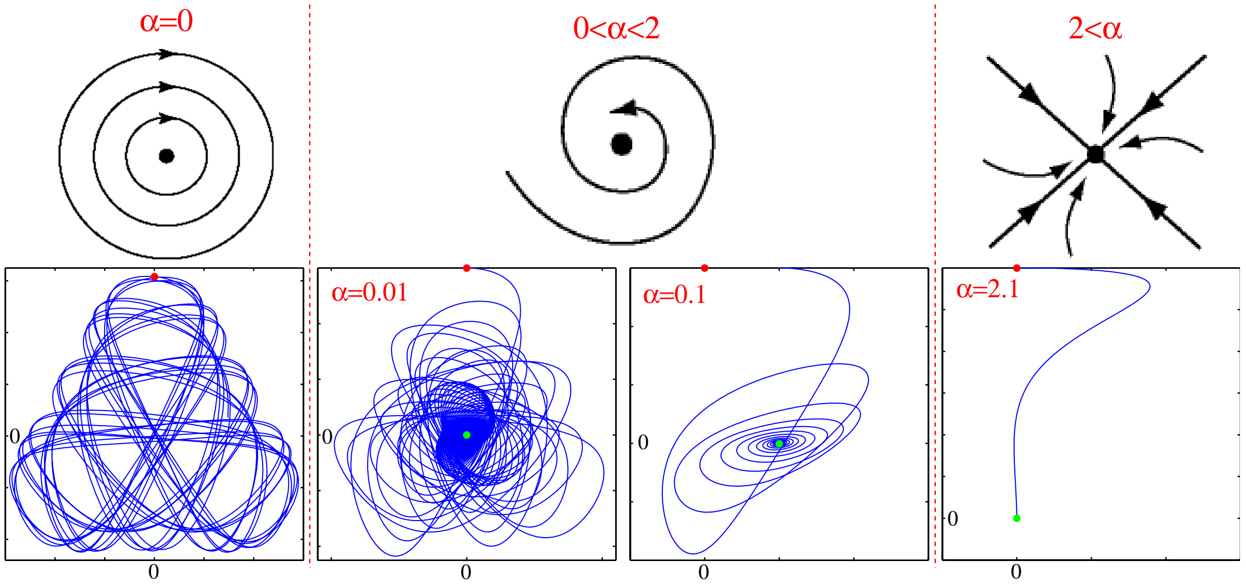


Fig. 5. Behavior of the equilibrium point at the origin and some illustrative orbits of initial energy $E = 0.15$.

of center type (see Fig. 5) for the Hamiltonian case $\alpha = 0$, becoming a stable focus for $0 < \alpha < 2$ and a stable node for $\alpha \geq 2$ (so, for $\alpha > 0$ is an attractor). For the dissipation values studied in this paper the equilibrium at the origin is a stable focus, and so the orbits fall to the origin following spirals (see Fig. 5 for examples).

Block (b) of Figs. 2 and 3 shows the physical space (x, y) for a dissipation of $\alpha = 0.01$ and $\alpha = 0.1$. For the energy value of $E = 0.15$ the OFLI2 plot shows that the chaoticity decreases with growing α because the orbits fall faster to the attractor (note that in this case almost all orbits are regular and the color code just remarks the transient behavior and the time to reach the attractor). For the other energy values, the basin plots show that as the dissipation grows the bounded region increases due to the attractor. As the energy grows, the escape basins are increased. In the plates we indicate the energy corresponding to the initial condition, as it may change due to the dissipation. We observe that for high values of the energy, for instance $E = 0.5$, and insofar we increase its value, the green region that represents the bounded orbits disappears because all particles escape. In this case, the exit basins are almost identical to the unperturbed case (as shown in the right column of Figs. 2 and 3) because the effect produced by the dissipative term is very small in comparison with the effect produced by the high energy of the system. The exit basins become identical when $E \to \infty$.

4. Noisy Case

The presence of noise is characteristic in several physical situations as in the transport and trapping of chemically or biologically active particles in large-scale flows [Motter *et al.*, 2003] in which noise is a natural ingredient. The study of the noise in open Hamiltonian systems have been carried out recently in [Seoane & Sanjuán, 2008; Seoane *et al.*, 2009]. In [Seoane & Sanjuán, 2008] the authors show that the basin structure in phase space is completely destroyed because the presence of noise even if its intensity is very small. For this reason, we cannot speak about the Wada property of the basins and the phase space appears smeared. The attractors are destroyed due to the noise effects and all trajectories diverge from the scattering region randomly. Figures 2(c) and 3(c) show the basins in both the phase space and the physical space. We clearly observed that also the physical space appears smeared when noise is present. If we increase the value of the energy, the effects of the noise disappear and the exit basins are very similar to the unperturbed case, as we have explained in the previous section for the dissipative case.

Another direct consequence of this fact is that in the presence of noise, particles tend to escape faster from the scattering region as compared with the noiseless case. The noise can render particle decay exponential [Seoane *et al.*, 2009]. This is the result of the role of the noise in the redistribution of the particles in the scattering region at different time steps. Figure 4(c) shows this exponential behavior.

5. Periodic Driving Case

The effect of a periodic driving in this kind of systems has not been much explored. The physical implications of this perturbation, such as in problems concerning the effect of companion galaxies, chaotic hamiltonian pumps, among others [Kandrup & Novotny, 2004; Kawai *et al.*, 2007; Hennig *et al.*, 2008; Zhang *et al.*, 2008; Chacón, 1994; Dittrich *et al.*, 2003], has motivated us to investigate its effects in the Hénon–Heiles system. For this purpose, we have fixed the value of the driving amplitude $A = 0.1$ (since, from the escaping point of view, its value plays the role to help/avoid that the particles eventually escape from the potential) and we have analyzed its effects for different values of the frequency ω. We have taken values of the frequency at $\omega = 1$, that correspond with the value of the resonant frequency in which $T = 2\pi$ that corresponds to the normal modes period for $E = 0$, and values of $\omega = 0.1$ (below the resonant value) and $\omega = 10$ (above the resonant value).

Figures 2(d) and 3(d) show plots of the basins in the (x, y) and $(y, \dot{y})$ plane for frequency values $\omega = 0.1$, $\omega = 1$ and $\omega = 10$. We observe the important role of the frequency in the topology of the basins. The symmetry of the basins has disappeared. This can be observed if we compare the case with energy $E = 0.15$ and $\omega = 10$ with the OFLI2 plot of the unperturbed case. Note that for $\omega = 0.1$ and $\omega = 1$ all the orbits are also escaping orbits of energy $E = 0.15$ but we show both, the OFLI2 and the exits basins, since OFLI2 also gives information about the transient and the exit times. On the other hand, for energy $E = 0.15$ and $\omega = 10$, the orbits never escape and therefore exit basins do not exist. This is the reason for which the exit basins, for $E = 0.15$ and $\omega = 10$, have not been plotted in Figs. 2 and 3. The behavior of the exit basins at high energies shows that they tend to equate each other, and the effect of the perturbation due to the periodic driving decreases as expected. Moreover, we see that the resonant case with the normal modes is, by difference, the most perturbed case for low energy values and also the low frequency case. In both situations, we see that the escaping orbits appear before the escape energy for the unperturbed case, and so, the periodic forcing gives a mechanism to force the orbits to escape. On the contrary, a high frequency perturbation just modifies slightly the system. Another consequence on the basins due to the effect of high values of the energy is the decrease of the fractality. This fact is shown in Figs. 2(d) and 3(d), in which we observe, as in the previous cases, the same question related to the unpredictability in the evolution of the system.

This result explains the behavior of the particles escaping from the scattering region as indicated as follows. We expect that, in the presence of a periodic forcing, the decay law becomes exponential where the best performance for the escapes takes place for $\omega = 1$. This phenomenon is illustrated in Fig. 4(d) where each curve corresponds to values of the frequency $\omega = 1$, $\omega = 0.1$ and $\omega = 10$, respectively. The particles escape faster from the scattering region for $\omega = 1$, which is in agreement with results presented in Figs. 2(d) and 3(d). Notice that the oscillations observed in the curve $\omega = 1$ of Fig. 4(d) are due to the effect of the frequency on the restoring force.

6. Conclusions and Discussion

We have analyzed the effects of different perturbations on both the topology and the escaping dynamics in the paradigmatic Hénon–Heiles system. We have characterized the basin topology associated to the physical (x, y) and phase $(y, \dot{y})$ spaces finding Wada basins for the periodic forcing case. When the orbits were bounded, we have computed the OFLI2 chaos indicator to distinguish between chaotic and stable orbits. Its behavior was observed to be different than for the unperturbed case. We have also addressed the effect of a periodic forcing in the survival probability of the particles in the scattering region. The best performance for the escapes are obtained for the case of the resonant frequency $\omega = 1$ that corresponds to an exponential decay law. Finally, we expect these results to be relevant in the field of open dynamical systems that have implications in problems of chaotic scattering which has applications in different areas of physics.

Acknowledgments

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