Partial control of chaotic transients using escape times

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Juan Sabuco[1], Samuel Zambrano and Miguel A F Sanjuán

Departamento de Física, Universidad Rey Juan Carlos, Tulipán s/n,
28933 Móstoles, Madrid, Spain
E-mail: juan.sabuco@urjc.es

Abstract. The partial control technique allows one to keep the trajectories of a dynamical system inside a region where there is a chaotic saddle and from which nearly all the trajectories diverge. Its main advantage is that this goal is achieved even if the corrections applied to the trajectories are smaller than the action of environmental noise on the dynamics, a counterintuitive result that is obtained by using certain *safe sets*. Using the Hénon map as a paradigm, we show here the deep relationship between the safe sets and the sets of points with different escape times, the *escape time sets*. Furthermore, we show that it is possible to find certain *extended safe sets* that can be used instead of the safe sets in the partial control technique. Numerical simulations confirm our findings and show that in some situations, the use of extended safe sets can be more advantageous.

Contents

[1] Author to whom any correspondence should be addressed.

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1. Introduction

One of the most important phenomena found in nonlinear dynamics is transient chaos [1]. The main characteristic of a system showing transient chaos is that almost all the trajectories passing through a square Q in the phase space behave chaotically for a while before eventually leaving it. The topological structure inside Q that causes this behaviour is a zero measure set known as a *chaotic saddle*. This situation arises in dynamical systems that act like a horseshoe map on a given square Q, a situation that is typically related to the existence of transverse homoclinic intersections [2], which are quite common in nonlinear dynamical systems.

There are several contexts in science and engineering where it would be desirable to make this transient chaotic behaviour permanent. In mechanics for example, the preservation of transient chaos can prevent the appearance of undesired resonances. In lasers, it has been shown that maintaining transient chaos can help one to avoid undesired intensity peaks. In engineering, it is known that the thermal pulse combustor operates chaotically, but when one tries to achieve high efficiency this can destroy the chaos and cause flameout. In population dynamics, the transition from transient chaos to periodicity is usually related to pathological situations (extinctions). These are examples that can be found in the techniques that have been proposed in recent years to achieve the goal of preserving transient chaos [3]–[9]. But two important issues must be addressed to solve this problem: one is the repulsive nature of the chaotic saddle and the other is the environmental noise present in many physical situations, which typically makes the orbits escape even faster (although in some cases, noise can slow down the escape process; see [10]).

Recently, an advantageous technique was proposed for this type of situation, referred to as the *partial control* technique [11, 12]. A remarkable achievement of the partial control technique is that it allows us to control the system even when the amplitude of the corrections applied to the trajectories (the *control*) is smaller than the maximum deviation of the trajectories from their deterministic path due to the presence of environmental noise (the *noise*). The basic ingredient for obtaining this somehow counterintuitive result is the use of certain sets referred to as *safe sets* [11] in Q. These sets have certain particular geometrical properties that are related to the typical stretching and folding action of the horseshoe-like mapping of Q. This advantageous control technique has been applied to well-known physical models such as an open billiard [11], the bouncing ball map [13] and a system with fractal basin boundaries [14]. This technique has also been used to make permanent chaotic some transient chaotic one-dimensional maps [15, 16]. In the case of a three-disc open billiard, this technique is used by applying small perturbations each time the trajectory of the systems hits one particular disc, which causes the trajectory to stay inside the billiard forever. An overview of the method can be found in [17].

In this context, an important question arises: Are there other sets inside Q that can play a role analogous to the safe sets in the partial control technique? In this paper we provide a positive answer to this question, by defining a generalization of the safe sets: the *extended safe sets*. We show here that these sets can be built using an algorithm discarding points from another important family of sets, the *escape time sets* inside Q, that corresponds to points in the square that escape from it under different numbers of iterations. We show here how these extended safe sets also allow one to keep trajectories inside Q with a control smaller than noise, and we discuss the advantages of using these types of sets. We also compare numerically the performance of the partial control technique using both safe sets and extended safe sets. An interesting result is

that the use of extended safe sets requires a smaller control than the use of safe sets for certain values of the noise, so the results of [11, 12] are partially improved. As we will discuss later, our results imply an important step forward in two interesting issues: the detection of extended safe sets in dynamical systems and its generalization to higher dimensional dynamical systems.

This paper is organized as follows. In section 2 we describe the basic situation in which our control strategy applies and we describe the system that we use in our explorations: the Hénon map. In section 3 we revise the partial control technique and describe the safe sets. In section 4 we define the escape time sets and explore their relation to the safe sets, and in section 5 we show the conditions that extended safe sets need to fulfil. In section 6 we explain how to use the extended safe sets in the partial control technique. Section 7 provides a numerical exploration of our control technique and a comparison between the results obtained with extended safe sets and safe sets. In section 8 we draw the main conclusions of this work.

2. Basic setting

We consider here dynamical systems of the form $p_{n+1} = f(p_n)$, where $p_n \in \mathbb{R}^2$. We assume that the map f acts on a square Q like a horseshoe map; for details see [12]. This implies that nearly all the trajectories inside Q (except a zero measure set) escape from it after iterations. On the other hand, the behaviour inside the square Q is erratic because of the existence of a zero measure nonattractive set, the chaotic saddle.

An example of this type of dynamical system is the Hénon map *with an adequate selection of the parameters*. The Hénon map is a paradigmatic system in the nonlinear dynamics community, and for that reason we have chosen it, from now on, to show how the partial technique works. The Hénon map is defined as

$$\begin{cases} x_{n+1} = a - by_n - x_n^2 & y_{n+1} = x_n. \end{cases} \tag{1}$$

For $a = 2.12$ and $b = 0.3$ we can obtain one of the most famous attractors, the Hénon attractor, as in figure 1(a), where its basin of attraction (with fractal boundary) and the basin of attraction of infinity are shown. *This is not the situation that we focus on here*. We are interested here in the situation where $a = 6$ and $b = 0.4$. The basin of attraction of infinity for these values of the parameters is shown in figure 1(b): we can see that all trajectories (except a zero measure set) diverge to infinity. This is due to the fact that the Hénon map f acts like a horseshoe map on the square $Q \equiv [-4, 4] \times [-4, 4]$, as shown in figure 2. The results that we have obtained are valid for maps acting on a square like a horseshoe map, i.e. satisfying the Conley–Moser conditions [12]. For these values of the parameters, the Hénon map satisfies these conditions, so we use it here both to illustrate and to numerically test our results.

In figure 2, we also show the two fixed points of this horseshoe-like map, p^* and p^{**}; the former will later play an important role (remember that every horseshoe map has associated with it two fixed points). The chaotic saddle responsible for transient chaos is shown in figure 3; it has been computed using DYNAMICS software [18]. Due to the horseshoe mapping, this set is topologically equivalent to an intersection of two Cantor sets of vertical and horizontal lines, as expected. Thus, *nearly all* the points inside the square (except a zero measure set, the chaotic saddle and its stable manifold) escape from it under iterations. The dynamics inside the set is chaotic, but due to its nonattracting nature for a typical trajectory starting inside Q we have transient chaos.
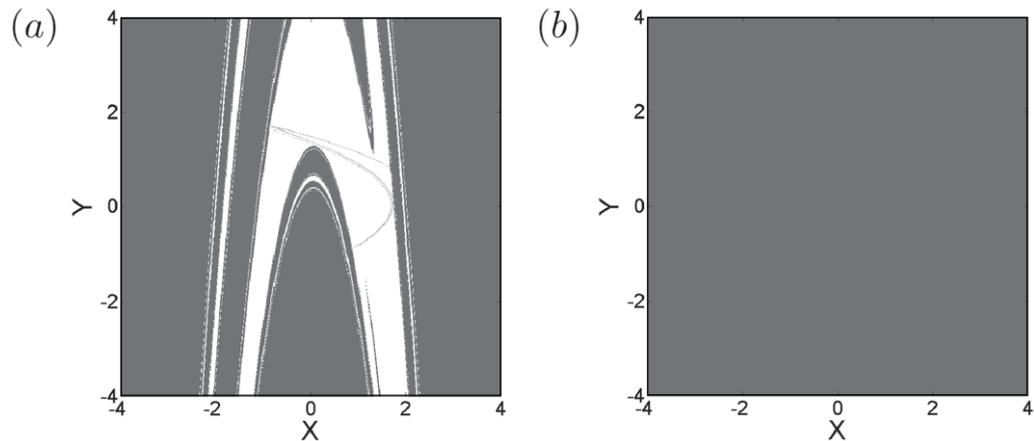
Figure 1. Basins of attraction for bounded attractors (white) and infinity (black) for the Hénon map: (a) $x_{n+1} = 2.12 - 0.3y_n - x_n^2$, $y_n = x_n$ where the attractor is shown in grey and (b) $x_{n+1} = 6 - 0.4y_n - x_n^2$, $y_n = x_n$, both in the square $Q = [-4, 4] \times [-4, 4]$. For the second set of parameters, nearly all the trajectories escape. The main goal of our control technique is to avoid those escapes from Q.
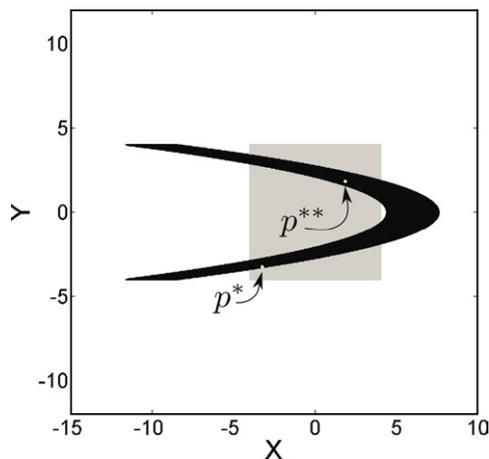

Figure 2. The square $Q = [-4, 4] \times [-4, 4]$ (grey) and its image under the Hénon map $x_{n+1} = 6 - 0.4y_n - x_n^2$, $y_n = x_n$ (black). These are the parameters used in the paper, for which the map acts like a horseshoe map on Q. The two saddle fixed points p^* and p^{**} are shown.

As we said before, we consider systems with this kind of escaping dynamics $p_{n+1} = f(p_n)$ and also affected by noise. This is modelled here by adding at each iteration a random perturbation $u_n \in \mathbb{R}^2$ that we refer to as the *noise*, which is bounded by the constant u_0, $\|u_n\| \leqslant u_0$. Thus, the system to be controlled is $p_{n+1} = f(p_n) + u_n$. The effect of noise is that *all* the trajectories inside the square Q will now escape from it under iterations. In order to test our results, we use here a noise with a uniform probability distribution, but the control technique has to work for any other kind of distribution.
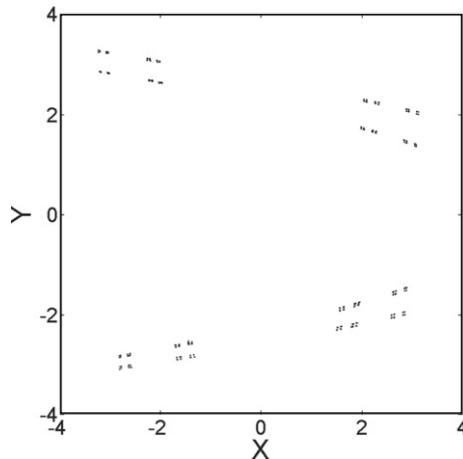
Figure 3. Chaotic saddle for the Hénon map $x_{n+1} = 6 - 0.4y_n - x_n^2$, $y_n = x_n$.

The control technique that we propose here in order to keep trajectories in Q is a targeting technique like other techniques proposed to sustain transient chaos, as in for example [8]. The idea behind this type of technique is that at each iteration, we apply an accurately chosen *control* $r_n \in \mathbb{R}^2$ to correct the system's trajectory considering the joint effect of the system's dynamics and the noise. The control is also bounded, so $\|r_n\| \leqslant r_0$. We can express this in a mathematical way using the formula

$$\{q_{n+1} = f(p_n) + u_n \qquad p_{n+1} = q_{n+1} + r_n. \tag{2}$$

This means that at each iteration the map and the noise act. Then, we apply a control r_n that corrects slightly the trajectory and places it in its new position, p_{n+1}. The technique that we describe here allows one to keep trajectories in an arbitrary square Q (if there is a horseshoe map) with $r_0 < u_0$.

Before describing the partial control technique we propose here, we note that different approaches can be used to keep p_n inside Q. A first approach would be to use the control r_n to steer the trajectories towards points inside Q with long-lived transient chaos (close to the chaotic saddle), as in [8]. Another approach [3] would be to use this control to steer the trajectories towards the stable manifold of one of the saddle fixed points, for example p^*. Simple geometrical considerations show that these two methods would allow one to keep the trajectories inside Q only if $r_0 \geqslant u_0$. This is somehow an intuitive result: in order to control an unstable system with noise, one might need a control at least equal to the noise.

Remarkably, the partial control technique that we describe below allows us to achieve this goal even if the control is smaller than the noise, that is, if $r_0 < u_0$. As will become clear later, this is due to the existence of certain sets inside the square Q: the *safe sets*. One of the main contributions of this paper will be to generalize these sets to some new extended safe sets.

3. The partial control technique and safe sets in a nutshell

As mentioned, the partial control technique was originally designed making use of certain safe sets. We describe this here in some detail. We call *vertical curves* the curves going from the top to the bottom of the square Q. We call S^0 the vertical curve (segment) that divides the square Q
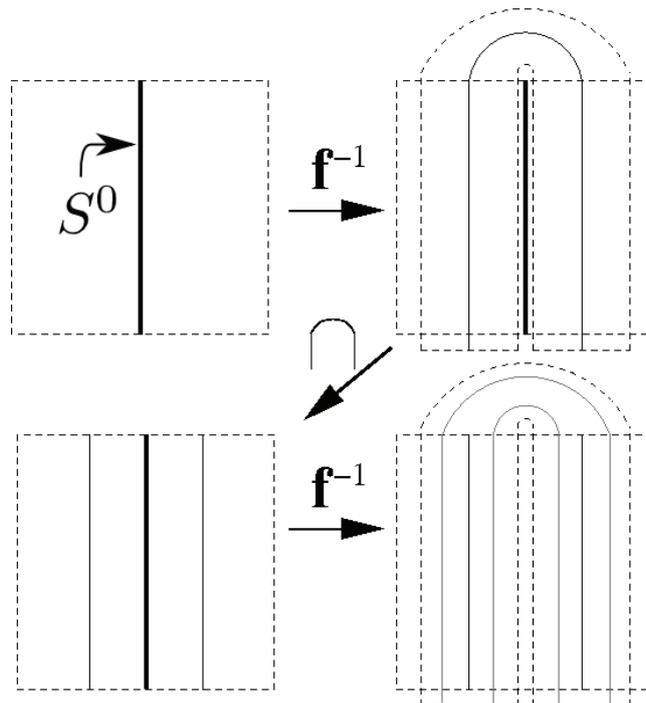
Figure 4. Basic action for generating inductively the safe sets in a map showing horseshoe-like behaviour in the phase space. We begin with a vertical segment (S^0) that splits the square into two equal halves and we compute the preimages of that line in Q. The intersections of the kth preimage of S^0 with Q gives the set S^k, which consists of 2^k curves.

into two halves, such that $f(S^0)$ is not on Q. The presence of a horseshoe map guarantees (for a general system) the existence of the safe sets S^k for $k = 1, 2, \ldots$ that are defined as

$$S^k = f^{-1}(S^{k-1}) \cap Q = f^{-k}(S^0) \cap Q. \tag{3}$$

The basic operation giving rise to the safe sets is illustrated in figure 4: the set S^k would be the preimage inside Q of S^{k-1}. In figure 5 the safe sets S^k from $k = 1$ to $k = 4$ are shown for the Hénon map. We can note that as k grows the sets get closer to the stable manifold of the invariant set inside Q. This is not surprising, provided that by definition, as k grows, points on S^k take a longer time to escape from Q. In this picture, it is also easy to see that the basic properties of the safe sets are the following [11]–[14, 16, 17]:

(i) S^k consists of 2^k vertical curves, which can be grouped in 2^{k-1} consecutive *pairs of curves* from left to right.

(ii) There is a curve of S^{k-1} between any pair of curves of S^k.

(iii) The maximum distance between any of the curves of the 2^{k-1} pairs of curves of S^k goes to zero as $k \to \infty$.

In order to properly describe the partial control technique, we need some extra definitions, which are illustrated in figure 6 for S^2. For each set S^k we define the middle curves ζ_i,
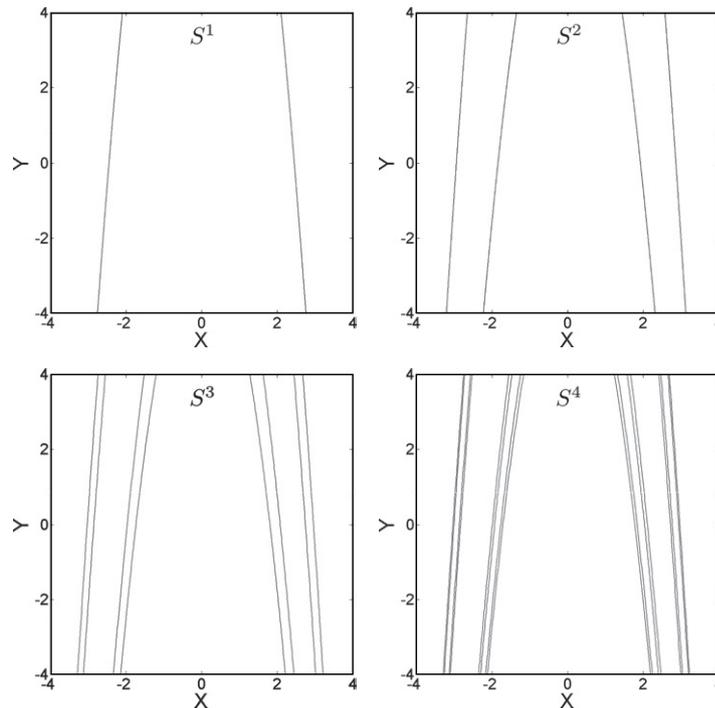
Figure 5. The sets S^1, S^2, S^3 and S^4 that are the result of computing the corresponding pre-image of S^0, which would be the vertical segment splitting the square into two equal rectangles. The set S^1 consists of a pair of curves, the set S^2 has two pairs of curves, the set S^3 consists of four pairs of curves and so on. Their geometrical properties are key to the application of the partial control technique.
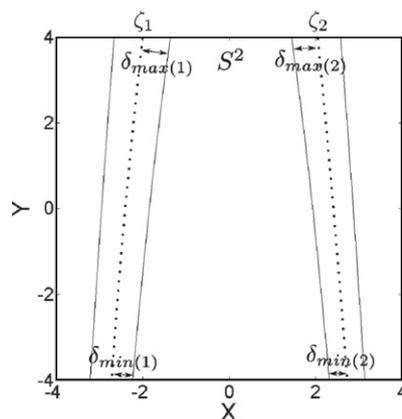

Figure 6. These are the parameters needed in the partial control technique using the S^2 sets. The curves ζ_1 and ζ_2 are the curves whose points are at the same distance from a curve of each pair of curves of S^2. We can also see that $\delta_{\max(i)}$ is the maximum distance from each pair of curves of S^2 to the curves ζ_i and $\delta_{\min(i)}$ is the minimum distance.
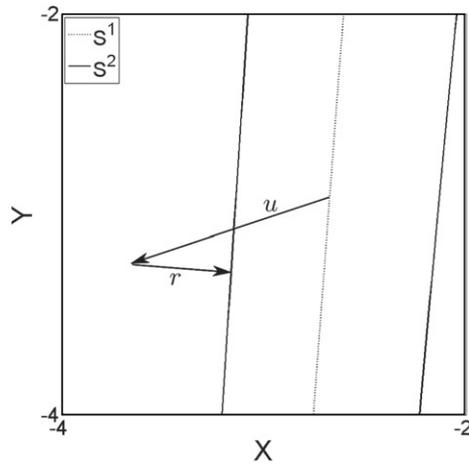
Figure 7. The basic idea of the partial control technique. The point p lies on S^2. If u_0 satisfies condition (4) on S^2, then $f(p)$ lies on a curve of S^1 that has two adjacent curves of S^2. Independently of the noise deviation u, a smaller control r can put the trajectory back on S^2.

$i = 1, \ldots, 2^{k-1}$, as the vertical curve equidistant to each of the curves of a pair of S^k. Then we call $\delta_{\max(i)}$ and $\delta_{\min(i)}$, $i = 1, \ldots, 2^{k-1}$, the maximum and minimum distances, respectively, between each of the curves of each pair and the corresponding middle curve ζ_i.

If we label $\delta_{\max}$ as the largest $\delta_{\max(i)}$ and $\delta_{\min}$ as the smallest $\delta_{\min(i)}$, then a partial control strategy for keeping trajectories bounded with a control smaller than noise can be obtained if

$$u_0 > \delta_{\max}. \tag{4}$$

With all these parameters defined we can finally introduce the partial control strategy [11]: assume that an iteration p lies on S^2 and u_0 satisfies the above condition for S^2. Then $f(p)$ will lie on a vertical curve of S^1, which has two adjacent curves of S^2 by (ii). The deviation induced by noise leads to $q = f(p) + u$. However, due to property (ii), no matter what the noise deviation is, a control such that $\|r\| \leqslant r_0$ with

$$r_0 = \max\{\delta_{\max}, u_0 - \delta_{\min}\} \tag{5}$$

can make the next iteration $p' = q + r$ lie again on S^2, and this can be repeated for each iteration. This is illustrated in figure 7. The control applied obviously satisfies $r_0 < u_0$ and trajectories are kept inside Q forever.

In order to find out if there are other sets in Q that can be used instead of safe sets to keep trajectories inside Q with $r_0 < u_0$, we introduce in the next section an important family of sets: the escape time sets.

4. Safe sets and escape time sets

Our aim now is to show the relationship between the safe sets and the *escape time sets* T^n. The escape time sets are the sets of points in a square with a horseshoe dynamics that stay inside the square under n iterations or more, that is,

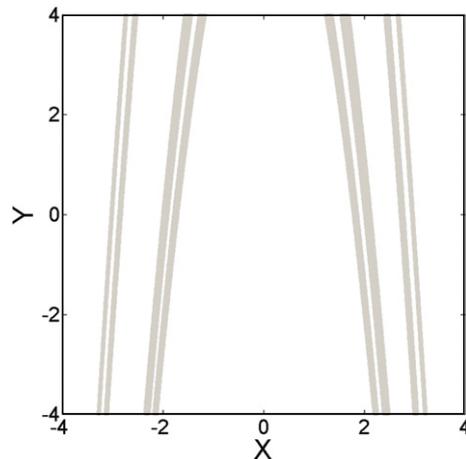$$T^n \equiv \{p \in Q / f^n(p) \in Q\} = f^{-n}(Q) \cap Q. \tag{6}$$

Figure 8. The escape time set T^3, i.e. the set of points that escape from Q after three or more iterations. It consists of four pairs of strips, and it is reminiscent of S^3.

By making some considerations on the inverse horseshoe map, it is not difficult to see that:

(i) T^n consists of 2^n vertical strips, which can be grouped in 2^{n-1} consecutive *pairs of strips* from left to right.

(ii) $T^m \subset T^n$ if $m > n$.

(iii) As the order n increases, the width of the strips of the T^n sets decreases.

The escape time set T^3 is shown in figure 8. It is clear that there is a resemblance between T^3 and S^3. In fact, the above properties are similar to those of the safe sets sketched in the previous section. Furthermore, it is easy to see from equations (3) and (6) that $S^n \subset T^n$ (provided that $S^0 \subset Q$), as shown in figure 9 for S^2 and T^2.

These similarities, however, do not imply that by using T^n instead of the sets S^n in the partial control technique, trajectories can be kept inside Q with $r_0 < u_0$. As we can see from figure 10 for T^2, the forward iterates of its four strips do not fall in the space between each pair of strips of T^2 (including them). Simple geometrical considerations using the ideas sketched in the previous section show that this implies that trajectories can be kept inside Q using T^2 instead of S^2 only if $r_0 > u_0$. Thus, some points on T^n need to be discarded, giving rise to the extended safe sets.

5. Obtaining extended safe sets from escape time sets

Considering this, clearly the escape time sets T^n are not a good substitute for safe sets S^n in the partial control technique. The next question would be: Which points on T^n have to be discarded so that the partial control strategy can be applied with $r_0 \leqslant u_0$? In this section we address this question. We describe here the *maximal extended safe set* E_{max}^n, which is a subset of T^n that can be used in the partial control technique so that trajectories can be kept bounded with $r_0 = u_0$. From these sets we can easily define the *extended safe sets* E^n, and for any E^n we show that there is a u_0 so that trajectories can be kept inside Q with $r_0 < u_0$.
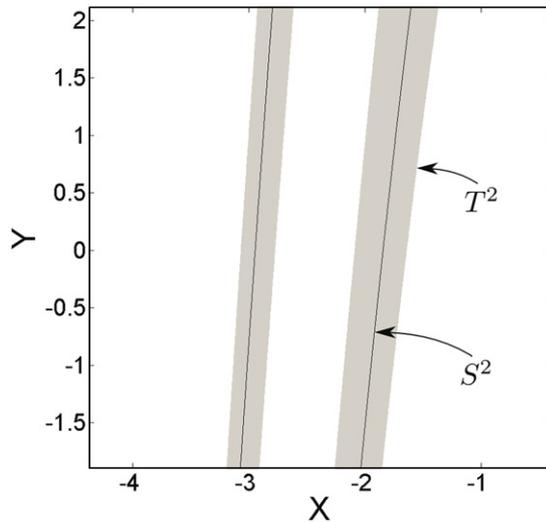
Figure 9. The black curves of this figure are points of the S^2 set, whereas the grey bars represent T^2. Here we can see how the safe sets S^2 are inside the space filled by T^2.
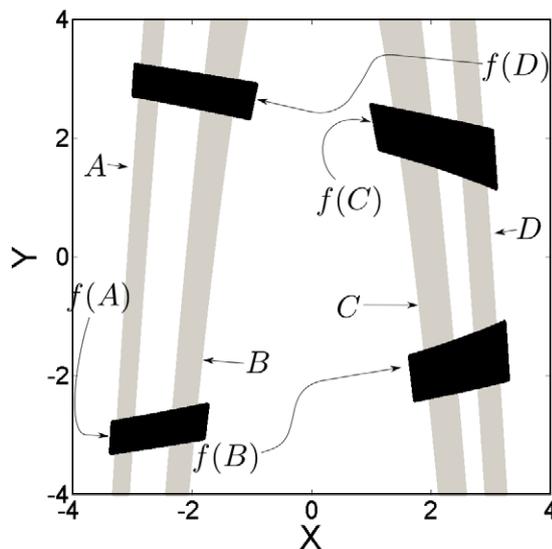

Figure 10. In this picture, we can see in grey the set of points of T^2 (the set of points that stay in the square under two or more iterations). We have also plotted in black the images of those points under one iteration. Because the images of the strips do not lie between the pairs of strips, it is impossible to use the set T^2 to keep trajectories inside Q with a control smaller than noise.

Recall the fixed point p^* shown in figure 2. We call $W^s(p^*)$ its stable manifold, and $W^s_L(p^*)$ the vertical curve that is a piece of $W^s(p^*)$ inside Q containing p^*. Consider now the set T^1 and the four vertical curves of $f^{-2}(W^s_L(p^*)) \cap Q$ shown in figure 11. We call $E^1_{\max}$ the set resulting from 'cutting' the two strips of T^1 into two thinner strips as these four curves indicate. The strips of $E^1_{\max}$ are mapped as shown in figure 11. This is due to the horseshoe mapping and to
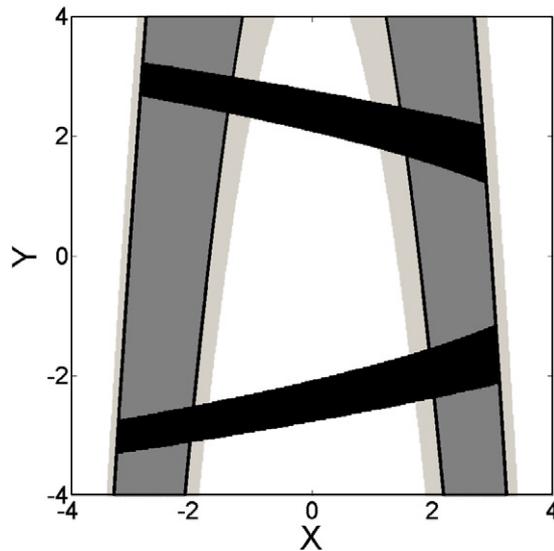
Figure 11. The set T^1 (light grey) is subdivided using four pieces of the stable manifold of p^* (black curve), so we obtain the maximal extended safe set $E^1_{\max}$ (grey). The image of $E^1_{\max}$ under f is shown (black); each of its two pieces falls in the space between the pair of strips of $E^1_{\max}$.

the fact that points in the stable manifold map into points of the stable manifold under f. This is the limit point of the 'good mapping' that we are searching for: the image of each strip of $E^1_{\max}$ falls into the space between the pair of strips of $E^1_{\max}$.

Considering this, we define inductively

$$E^{n+1}_{\max} = f^{-1}(E^n_{\max}) \cap Q = f^{-n}(E^1_{\max}) \cap Q. \tag{7}$$

Clearly the set $E^n_{\max}$ consists of 2^n strips, which also can be grouped in 2^{n-1} pairs of strips from left to right. Note that by definition it will be contained in T^n. Furthermore, it can be seen that the curves that bound each vertical strip of $E^n_{\max}$ are pieces of the stable manifold (since preimages of points of the stable manifold also belong to the stable manifold). These sets will reproduce the good kind of mapping observed for $E^1_{\max}$: the image of each strip of $E^n_{\max}$ falls in the space between each pair of strips of $E^n_{\max}$. This is shown for example in figure 12 for the set $E^2_{\max}$. Using the geometrical considerations provided above, we can see that by using the sets $E^n_{\max}$ instead of S^n, the partial control strategy trajectories can be kept inside Q with $r_0 = u_0$.

With these elements in mind, we can define the *extended safe sets* E^n as follows: an extended safe set E^n is a set of 2^n vertical strips, each of them inside a different strip of $E^n_{\max}$, so that their vertical bounds do not intersect with the vertical bounds of the strips of $E^n_{\max}$. Thus, an extended safe set is obtained when the width of all the strips of the extended safe sets is reduced. If we take a zero-width strip we would obtain safe sets as the ones described in section 3. An example of an extended safe set E^2 obtained from $E^2_{\max}$ is shown in figure 13. In the next section, we show that applying the partial control technique with an extended safe set it is possible to keep trajectories bounded with $r_0 < u_0$.

Before concluding this section we would like to point out that the same procedure that has been carried out with the Hénon map to obtain the extended safe sets could be repeated with the same level of difficulty for any dynamical system topologically equivalent to it, i.e. a dynamical
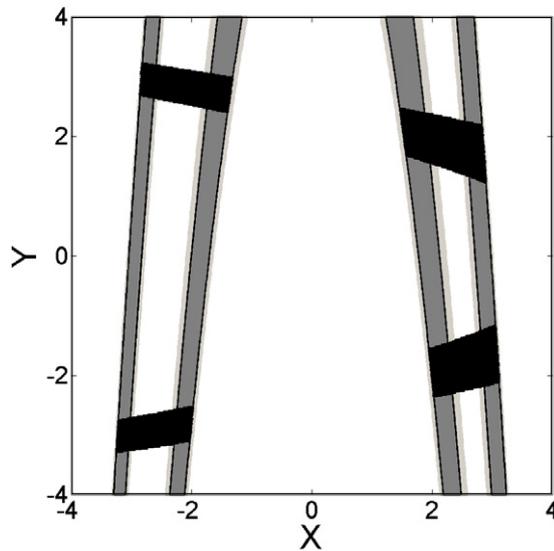
Figure 12. The set T^2 (light grey) is cut using eight pieces of the stable manifold of p^* (black curve) and gives rise to the maximal extended safe set E^2_{max} (grey). The images of each strip of E^2_{max} under f are shown (black); they fall in the space between each pair of strips of E^2_{max}.
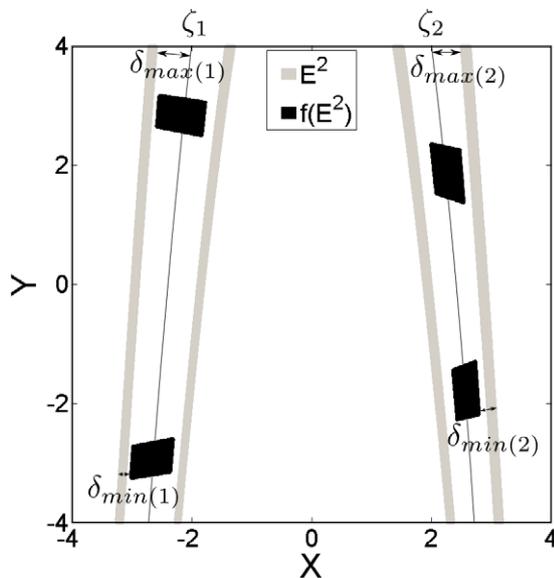

Figure 13. In this figure we can see an extended safe set E^2 and the parameters needed in the partial control technique using the extended safe set E^2.

system acting as a horseshoe on a (topological) square Q. It is important to note that in order to find the extended safe sets one needs to find the chaotic saddle, a square enclosing it, the escape time sets and the stable manifold of the fixed point of the horseshoe. These can be calculated using time series of the system, and do not require us to know exactly the form of the map f.

Thus, we consider that this is the main advantage of using extended safe sets from the point of view of applicability.

6. Partial control with extended safe sets

Now that we have described the extended safe sets E^n, we need some parameters to describe how they can be used in the partial control technique. For a set E^n the curves ζ_i, $i = 1, \ldots, 2^{n-1}$, represent the curve equidistant to any pair of strips. We define the maximum distance $\delta_{\max(i)}$ from ζ_i to the inner part of the pairs of strips for $i = 1, \ldots, 2^{n-1}$. Analogously, we define the minimum distance $\delta_{\min(i)}$ from the forward iteration of E^n to the outer part of the closer pair of strips (between which it has fallen) for $i = 1, \ldots, 2^{n-1}$. We can see all of them illustrated in figure 13.

If we label as $\delta_{\max}$ the largest $\delta_{\max(i)}$ among all the pairs of strips, we find that the condition required for the extended safe sets to have a control smaller than noise is the same as that for the classical safe sets

$$u_o > \delta_{\max}. \tag{8}$$

The maximum control needed can also be computed in the same way if we consider $\delta_{\min}$ as the smallest of all the $\delta_{\min(i)}$ among all the pairs of strips

$$r_0 = \max\{\delta_{\max}, u_0 - \delta_{\min}\}. \tag{9}$$

The partial control strategy [11] is similar to the one described for the safe sets. Assume that p lies on E^2 and u_0 satisfies the above condition. Then $f(p)$ will lie on a strip of E^1, which now falls in the space between two strips of E^2. The deviation induced by noise leads to $q = f(p) + u$. Now, no matter what the noise deviation is, again a control such that $\|r\| \leqslant r_0$ with

$$r_0 = \max\{\delta_{\max}, u_0 - \delta_{\min}\}, \tag{10}$$

obviously satisfying $r_0 < u_0$, can make $p = q + r$ lie again on E^2, and this can be repeated forever. This is illustrated in figure 14. In the next section, we carry out a numerical exploration of these results.

7. Numerical results

Here we show some numerical simulations showing the performance of the partial control technique with extended safe sets in comparison with the partial control technique with safe sets. In our numerical simulations we have used the sets E^1, E^2 and E^3 shown in figure 15.

In figure 16 we show an example of the control needed when stabilizing an orbit on E^3 with a maximum amplitude of noise of $u_0 = 0.25$. In this figure we show the amplitude of the control $\|r_n\|$ used in 1000 iterations of the map. As we can see for all the iterations, it is satisfied that $\|r_n\| < u_0$, i.e. the control applied is smaller than the noise, as claimed.

In order to compare the performance of the method using both extended safe sets and safe sets, we show in figure 17 a plot of the maximum control needed using E^3 and S^3 for different values of noise. This maximum control r_0 is estimated as the maximum value of the control applied $\|r_n\|$ when using our technique for a large number of iterations.
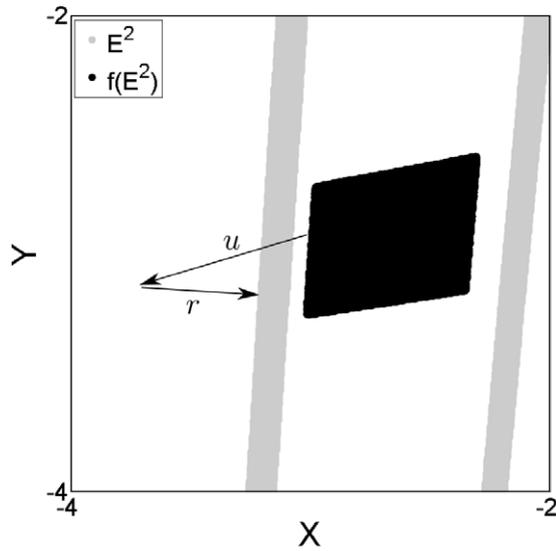
Figure 14. The basic idea of the partial control technique. If p lies on E^2 and u_0 satisfies condition 4 on E^2, then $f(p)$ lies on a strip of E^1 that has two adjacent strips of E^2. Independently of the noise deviation u, a smaller control r can put the trajectory back on E^2.
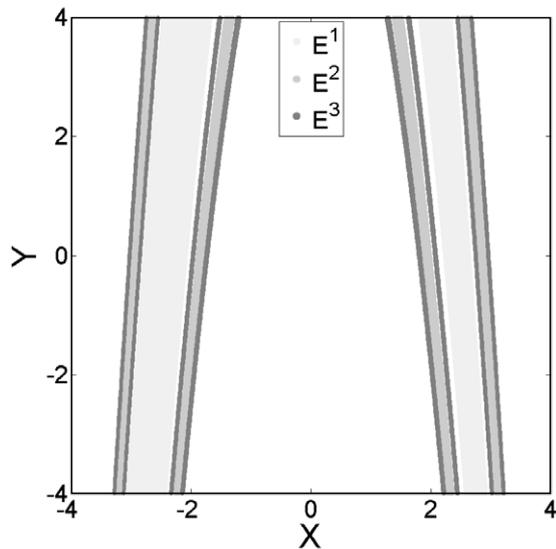

Figure 15. The extended safe sets used in our numerical simulations. They can play a role analogous to the safe sets in the partial control technique, so trajectories can be kept inside the square Q with a control smaller than noise.

Figures 17 and 18 illustrate well some of the basic features of the method. First we can note that the computed values of r_0 for the extended and safe sets are smaller than u_0 except for some ranges of u_0. This means that in those ranges the requirements of equation (8) are not fulfilled. For example, we can see in figure 18 that there is a u_0 interval for which the condition $r_0 < u_0$ is not fulfilled for S^2 or E^2, but we can see that the condition is fulfilled for S^3 and E^3. This
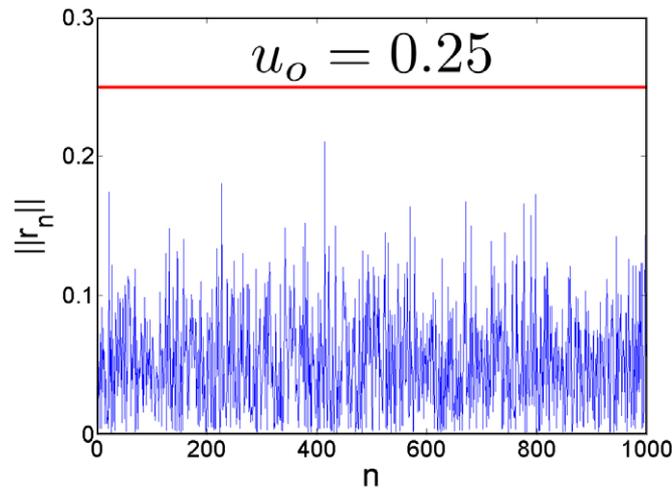
Figure 16. Control needed in 1000 iterations of the Hénon map where the maximum amplitude of noise is $u_0 = 0.25$. We can check that the amplitude of the control applied in each iteration $\|r_n\|$ is smaller than the noise.
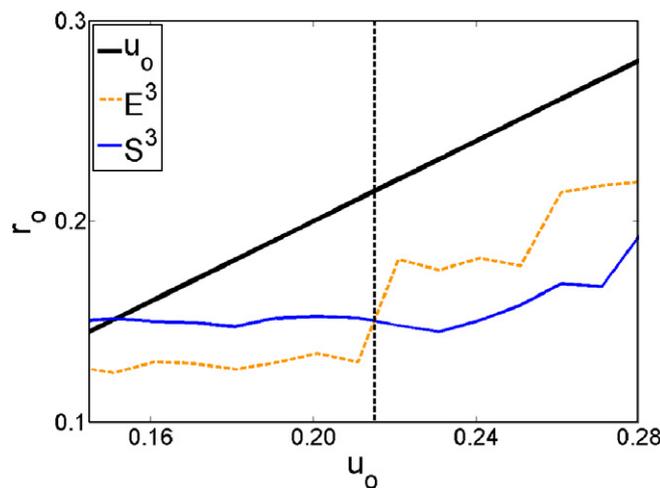

Figure 17. Necessary amplitude of the control r_0 needed when using the safe set S^3 and an extended safe set E^3. The value of r_0 is estimated by computing the maximum value of the amplitude of the control needed, $\|r_n\|$, to keep the trajectories inside Q. As we can see there is an interval where it is more effective to use E^3, after which it becomes more effective to use S^3.

illustrates the property by which, for all values of u_0, if k is sufficiently large, trajectories can be kept on E^k (or S^k) with $r_0 < u_0$. In other words, there is always a k for which the prescribed extended safe set E^k will fulfil equation (8).

We can also see that there exists a transition point that indicates the limit where the use of our extended safe set becomes less efficient than using the safe set. An explanation of this fact is beyond the scope of this paper, but the key idea is that for some values of the noise u_0, the fact that the extended safe sets are 'thicker' than the safe sets can be an advantage, whereas for some other values of u_0 it might be a drawback. In figure 18 we can see a more global comparison of
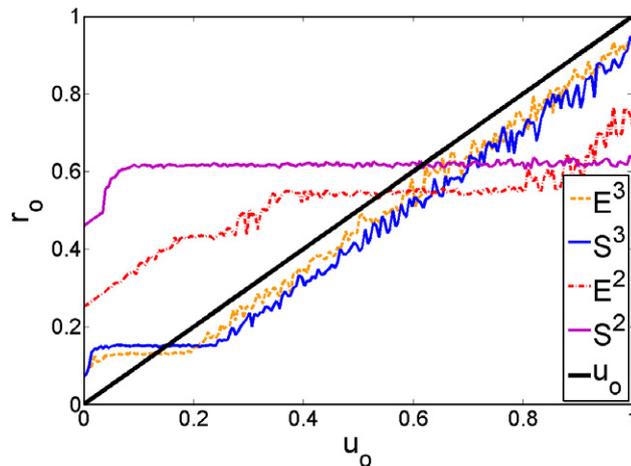
Figure 18. Here we show the necessary amplitude of the control r_0 needed when using the safe sets for a wider range of values of u_0. Here we can check that around levels of noise between 0.14 and 0.19, E^3 is the best option. From 0.19 to 0.65, S^3 requires a smaller control. We should use E^2 from 0.65 to 0.85. And we should start using S^2 for values of noise beyond 0.85.
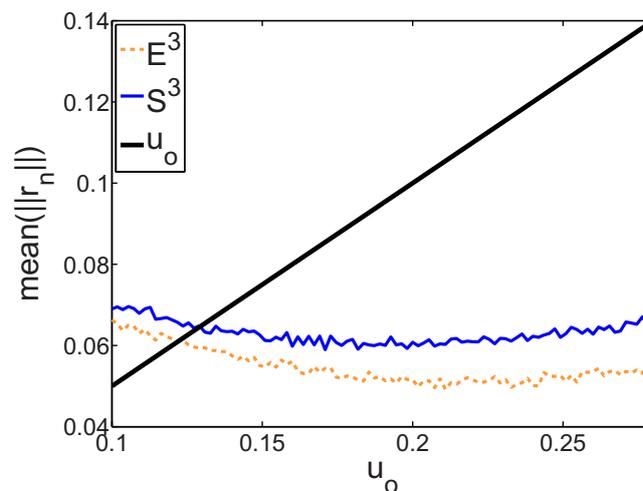

Figure 19. Another kind of comparison between the safe sets and the extended safe sets using the average value of control over long runs of iterations. Here for each value u_0 2000 iterations of the technique have been carried out and the mean value of all those iterations plotted on the figure.

both techniques using different safe sets and extended safe sets, which clearly shows that there are intervals where using extended safe sets requires a smaller control than using safe sets and that there is always an extended safe set such that trajectories can be kept inside the square with $r_0 < u_0$.

Finally, we have performed a simulation in order to compare both techniques in terms of the average of the control applied over a long number of iterations. This is what we can see

in figure 19, where it is possible to see that the average value of the control needed for the E^n sets is smaller than for the safe sets S^n for all the values of noise. This can be easily understood provided that the extended safe sets are 'thicker', so typically a smaller perturbation is needed to steer trajectories towards them.

8. Conclusions and discussion

In this paper, we have shown that it is possible to apply the partial control technique using certain sets, the extended safe sets, which are deeply related to the escape time sets, in a square Q where a horseshoe map exists. The notion of extended safe sets generalizes the notion of safe sets when we use these new sets in the partial control strategy, to keep trajectories bounded in the square, with a control smaller than the noise. The procedure for obtaining extended safe sets from escape time sets has been described and it implies the cut of the escape times using the stable manifold of a saddle fixed point in a horseshoe map (remember that every horseshoe map has associated with it two fixed points).

We have carried out numerical simulations testing this new technique and showing that it achieves our desired goal of control smaller than noise. We have also compared the performance of the partial control technique using safe sets and extended safe sets, finding that the best choice depends on the amplitude of the noise present in the system. We have also shown that over a long series of iterations, the mean of the control applied is always smaller when using extended safe sets than when using safe sets.

From an experimental point of view, the use of extended safe sets, being a nonzero measure set, is advantageous. Having an area, it is easier to place trajectories on them without making an error than when dealing with zero measure sets as the safe sets. On the other hand, by construction they are computed from the escape time sets and using the stable manifold of a fixed point as a guide, and this information can be inferred from time series of the system. Finally, we believe that the use of escape time sets might yield extensions of this technique to more general settings, such as for example to the control of nonhyperbolic transient chaos.

Throughout this paper we have assumed that our control has no errors, that is, for each iterate, we can place the trajectory exactly where we want. However, this is not a critical assumption. As with safe sets, it is possible to keep the condition $r_0 < u_0$ using extended safe sets also if we have small control errors [12], i.e. even if at each iteration trajectories are not placed exactly on the extended safe sets. This tolerance to errors depends basically on the value of r_0 needed in the absence of errors, the value of u_0 and the expansiveness of the map f, which somehow tells us how much we are penalized if we do not apply exactly the required control. To provide an analytical estimate of such tolerance, however, is complicated. However, due to the fact that extended safe sets are 'thicker' than safe sets, we expect that the tolerance for the former is bigger.

In spite of the advantages of our technique, there are still some limitations. In the current form of the technique it is required that all the variables can be measured and controlled, which might be a strong requirement in some situations, especially in experiments. However, it is possible to see how this can be achieved. First we can note that in the present technique, owing to the form of the extended safe sets, corrections would typically be applied only in the x-direction, generally speaking in the 'unstable' direction of the system. This suggests that the number of variables that need to be controlled for our control technique can be reduced. On the other hand, the key ingredient of our technique, the horseshoe mapping, is a topological

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feature of the system and thus it should be possible to reproduce it using embedding of an observable in a proper phase space with delay. Further work on this topic should provide a rigorous answer to these important issues.

Finally, we want to emphasize that our analysis reveals the deep relationship existing between the escape time sets and these extended safe sets, so we believe that any algorithm implemented in order to detect extended safe sets should make use of this relationship: first searching for the different escape time sets and then discarding the points that are not useful. As we said, this can be helpful both for an experimental detection of sets that can allow us to obtain partial control with $r_0 < u_0$ and in generalizations of the partial control technique to dynamical systems in higher dimensions.

Acknowledgments

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