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DETECTING DETERMINISM IN TIME SERIES WITH ORDINAL PATTERNS: A COMPARATIVE STUDY

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Dedicated to the memory of Valery S. Melnik

Detecting determinism in univariate and multivariate time series is difficult if the underlying process is nonlinear, and the noise level is high. In a previous paper, the authors proposed a method based on observable ordinal patterns. This method exploits the robustness of admissible ordinal patterns against observational noise, and the super-exponential growth of forbidden ordinal patterns with the length of the patterns. The new method compared favorably to the Brock–Dechert–Scheinkman independence test when applied to time series projected from the Hénon attractor and contaminated with Gaussian noise of different variances. In this paper, we extend this comparison to higher fractal dimensions by using noisy orbits on the attractors of the Lorenz map, and the time-delayed Hénon map. Finally, we make an analysis that enlightens the robustness of admissible ordinal patterns in the presence of observational noise.

Keywords: Ordinal patterns; permutation entropy; independence test; BDS test.

1. Introduction

Permutation entropy has attracted much attention among the researchers in time series analysis since it was introduced by Bandt and Pompe [2002]. In short, permutation entropy replaces the probabilities for symbol blocks in the outputs of a data source by the probabilities of the corresponding ordinal patterns, in the formula of Shannon entropy. An ordinal pattern of length $L \geq 2$, or ordinal L-pattern, is a permutation π of the numbers $\{0, 1, \ldots, L-1\}$, that will be written as $\pi = \langle \pi_0, \pi_1, \ldots, \pi_{L-1} \rangle$; the set of all $L!$ ordinal L-patterns will be denoted by $\mathcal{S}_L$. Given a length-L symbol block $x_k^{k+L-1} = x_k x_{k+1} \cdots x_{k+L-1}$, where the symbols (or 'letters') x_n are drawn from a finite and linearly ordered alphabet $S = \{s_1, s_2, \ldots, s_{|S|}\}$, we say that x_k^{k+L-1} defines (realizes, is of type, ...) $\pi \in \mathcal{S}_L$ if

$$x_{\pi_0} < x_{\pi_1} < \cdots < x_{\pi_{L-1}}.$$

In the case that $x_{\pi_i} = x_{\pi_j}$, we agree that $x_{\pi_i} < x_{\pi_j}$ if and only if $i < j$ (other conventions are, of course, possible). Notice that real-world and computer-generated data belong to finite alphabets on account of the finite precision of observation devices and real-number representation in computers. This setting extends to multivariate time series, the data being now, say, lexicographically ordered.

Permutation entropy can also be extended to dynamical systems in metric [Bandt & Pompe, 2002; Amigó *et al.*, 2005] and topological [Bandt *et al.*, 2002; Amigó & Kennel, 2007] versions. Interestingly enough, it can then be shown that, under certain mathematical assumptions, the orbits generated by the dynamics cannot realize all possible ordinal patterns. That is, there are always ordinal patterns of sufficient length, such that they cannot occur in any orbit whatever the initial condition. The existence of these "forbidden" ordinal patterns makes a difference between randomly (and without constraints) generated sequences, in which any ordinal L-pattern has probability one to occur, and deterministically generated sequences. The robustness of forbidden ordinal patterns against observational noise and their super-exponential growth rate with length [Amigó *et al.*, 2008a], make them a practical tool to detect dependence even in very noisy time series [Amigó *et al.*, 2007; Amigó *et al.*, 2008b].

The method proposed by the authors in [Amigó *et al.*, 2008b], for detecting determinism in noisy time series, calls for counting visible ordinal patterns and performing a subsequent chi-square test, the null hypothesis being that the data are white noise (i.e. outcomes of an independent and identically distributed process). The rationale for such a null hypothesis is that, as usual in physics, we mean by determinism that the data of a random-looking sequence are actually not independent. In order to check the quality of this forbidden pattern-based approach, we compared it with the Brock–Dechert–Scheinkman (BDS) test [Brock *et al.*, 1996] using series numerically generated by the Hénon map, whose attractor has fractal dimension $D_0 = 1.28 \pm 0.01$ [Sprott, 2006]. The BDS statistic was chosen because it has become a benchmark for independence testing in nonlinear series. In this paper we extend this comparison to the Lorenz map ($D_0 \geq 2$) and the family of time-delayed Hénon maps [Sprott, 2006], which provide attractors with high fractal dimensions.

This paper is organized as follows. In order to make this paper as self-contained as possible, we review briefly the theoretical background in Sec. 2. In the subsequent sections, the context, method and benchmark are presented. The results of the numerical simulations are illustrated in Sec. 5 with orbits on attractors with approximated fractal dimensions 2.0, 10 and 20. The actual study comprised of several cases more than those presented in Sec. 5, also different types of additive noise were considered, but the results were similar in all cases. In Sec. 6 we clarify the robustness of ordinal patterns against observational noise by means of a simple example. In the last section, the main conclusions are summarized.

2. The Theory

Suppose that the univariate time series $x_0^{N-1} = x_0, x_1, \ldots, x_{N-1}$ has been generated by a deterministic map, say, $f : I \to I$, where I is a one-dimensional closed interval. That is, $x_{n+1} = f(x_n) = f^n(x_0)$, where $f^0(x) := x$, and $f^n(x) := f(f^{n-1}(x))$ for $n \geq 1$. Then, we say that $x_0 \in I$ realizes the ordinal L-pattern $\pi = \langle \pi_0, \pi_1, \ldots, \pi_{L-1} \rangle$ if the initial segment x_0^{L-1} is of type π, i.e.

$$f^{\pi_0}(x_0) < f^{\pi_1}(x_0) < \cdots < f^{\pi_{L-1}}(x_0).$$

Ordinal patterns that are realized by an $x_0 \in I$, are called *allowed* or *admissible patterns* for f. Otherwise, they are called *forbidden patterns* for f. Allowed ordinal patterns are the main ingredient of the permutation entropy of a map. For brevity, we will refer only to the properties of ordinal patterns that follow from theoretical results on permutation entropy.

A one-dimensional interval map f is said to be piecewise monotone if there is a partition of the interval into subintervals such that f is continuous and strictly monotone on each of those subintervals. In this case, it can be proved (based on the concept of topological permutation entropy) that

$$|\{\pi \in \mathcal{S}_L : \pi \text{ is realized by } x \in I\}| \sim e^{Lh_{\text{top}}(f)}$$

where $|\cdot|$ denotes cardinality, $\sim$ stands for "asymptotically when $L \to \infty$", and $h_{\text{top}}(f)$ is the topological entropy of f. Since, on the other hand, $|\mathcal{S}_L| = L!$ grows super-exponentially with L (according to Stirling's formula), we conclude that, at variance with random sequences, there always exist ordinal patterns of sufficient length L that are forbidden in deterministic time series, and moreover, their number grows super-exponentially with L.

The theoretical situation in higher dimensions is less satisfactory in the sense that the existence of forbidden patterns has been proved so far only under the somewhat restrictive condition of expansiveness [Amigó & Kennel, 2007]. The baker map $B : [0,1]^2 \to [0,1]^2$, defined as

$$B(x,y) = \begin{cases} \left(2x, \dfrac{1}{2}y\right), & 0 \leq x \leq \dfrac{1}{2}, \\ \left(2x-1, \dfrac{1}{2}y + \dfrac{1}{2}\right), & \dfrac{1}{2} \leq x \leq 1, \end{cases}$$

provides a simple example of an expansive two-dimensional interval map, hence with forbidden patterns. Indeed, B is order-isomorphic (up to a set of measure 0) to the two-sided Bernoulli shift on two symbols, both the unit square $[0,1]^2$ and the space of binary bisequences being endowed with lexicographical order. Since the latter can be shown [Amigó *et al.*, 2008a] to have forbidden patterns of length $L \geq 4$, the same happens to B. In particular, it follows that the ordinal 4-patterns $\langle 0,2,3,1 \rangle$, $\langle 1,0,2,3 \rangle$, $\langle 2,0,1,3 \rangle$ and their "mirrored patterns" $\langle 1,3,2,0 \rangle$, $\langle 3,2,0,1 \rangle$, $\langle 3,1,0,2 \rangle$ are forbidden for the baker map. Moreover, numerical simulations with multidimensional maps support the claim that the existence of forbidden patterns is a general feature of deterministic multivariate time series.

An interesting property of forbidden patterns is the fact that each single one elicits a trail of longer forbidden patterns, called *outgrowth forbidden patterns*. Indeed, as noted in [Amigó *et al.*, 2006] together with the forbidden pattern $\pi = \langle \pi_0, \pi_1, \ldots, \pi_{L-1} \rangle$ all longer patterns of the form

$$\langle *, \pi_0 + n, *, \pi_1 + n, *, \ldots, *, \pi_{L-1} + n, * \rangle \in \mathcal{S}_N, \tag{1}$$

$(N > L)$ are also forbidden for f. Here $n = 0, 1, \ldots, N-L$, where $N - L \geq 1$ is the number of wildcards $* \in \{0, 1, \ldots, n-1, L+n, \ldots, N-1\}$ (with $* \in \{L, \ldots, N-1\}$ if $n = 0$ and $* \in \{0, \ldots, N-L-1\}$ if $n = N-L$). If $\mathcal{S}_N^{\text{out}}(\pi)$ denotes the family of length-N outgrowth patterns of $\pi \in \mathcal{S}_L$, then it can be proven [Amigó *et al.*, 2008a] that there exist constants $0 < c, d < 1$ such that $(1-d^N)N! < |\mathcal{S}_N^{\text{out}}(\pi)| < (1-c^N)N!$ Therefore, the number of outgrowth forbidden patterns grows super-exponentially with the length.

Other instrumental property of ordinal patterns for time series analysis is their robustness against observational noise. In the case of univariate series, this robustness follows readily from the fact that ordinal patterns are defined by inequalities. In the numerical simulations below, we project orbits belonging to higher-dimensional attractors, on their first coordinate, hence we will effectively deal with univariate sequences. Yet, this property alone would not explain the persistence of forbidden ordinal patterns in deterministic sequences perturbed with high levels of noise, as shown below and in [Amigó *et al.*, 2008b]. Indeed, in deterministic sequences there is a second, more important mechanism in place for the said robustness, also in higher dimensions: the dynamics itself. This point will be illustrated in Sec. 6 with the logistic map.

3. The Method

Our method to discriminate random from deterministic sequences builds on the existence of forbidden ordinal patterns in the latter. However, the implementation of this simple fact in practice has to overcome two major challenges: finiteness of the time series and observational noise. Finiteness entails *false forbidden patterns*, i.e. ordinal patterns missing in a finite random sequence just by chance [Amigó *et al.*, 2007]. Observational noise can destroy true forbidden patterns, although it can create also new forbidden patterns. We expect though that the robustness of ordinal patterns against observational noise and their proliferation with increasing length will make the difference.

Consider a time series of the form

$$\xi_n = f^n(x_0) + w_n, \tag{2}$$

$(0 \leq n \leq N-1)$ where w_n is white noise, i.e. outcomes of an independent and identically distributed (i.i.d.) random process. In order to tell white noise from a noisy deterministic (univariate or multivariate) time series of the form (2), we have proposed in [Amigó *et al.*, 2008b], a chi-square test based on the count of observable or "visible" ordinal patterns. The null hypothesis reads:

$$H_0: \text{ The } \xi_n \text{ are i.d.d.}$$

With this objective, take sliding windows of length $L \geq 2$, overlapping at a single point (i.e. the last point of a window is the first point of the next one) down the sequence $\xi_0^{N-1} = \xi_0, \ldots, \xi_{N-1}$. For brevity, we will call them "nonoverlapping" windows (although they do overlap at the endpoints). The number of such windows is

$$K = \left\lfloor \frac{N-1}{L-1} \right\rfloor, \tag{3}$$

each comprising the entries

$$\eta_k := \xi_{kL-k}, \ldots, \xi_{(k+1)L-(k+1)}, \quad 0 \le k \le K-1.$$

Notice that if the values $\xi_0, \xi_1, \ldots, \xi_{N-1}$ are independently drawn from the same probability distribution, then the ordinal L-patterns realized by the components of $\eta_k \in \mathbb{R}^L$, which we denote by $\pi(\eta_k) \in \mathcal{S}_L$, will also be independent and, moreover, be uniformly distributed random variables. Therefore, if one or several ordinal patterns of length L are missing in a sample obtained using nonoverlapping windows, this might be a statistically significant signal that independence and/or the equality of the distribution are/is not fulfilled.

Given the realization $\{\eta_k \in \mathcal{S}_L : k \ge 0\}$ corresponding to an arbitrarily long time series $\{\xi_n : n \ge 0\}$, suppose that some ordinal patterns of length L are missing in the initial segment $\xi_0, \xi_1, \ldots, \xi_{N-1}$. Let ν_j be the number of η_k's such that η_k is of type π_j (i.e. $\pi(\eta_k) = \pi_j \in \mathcal{S}_L$), $1 \le j \le L!$ Thus, $\nu_j = 0$ means that the pattern π_j has not been observed.

In order to accept or reject the null hypothesis H_0, Eq. (3), based on our observations, we apply a chi-square goodness-of-fit hypothesis test with the statistic [Amigó *et al.*, 2008b]:

$$\chi^2 = \sum_{j=1}^{L!} \frac{\left(\nu_j - \dfrac{K}{L!}\right)^2}{\dfrac{K}{L!}} = \frac{L!}{K} \sum_{j:\pi_j \text{ visible}} \nu_j^2 - K.$$

If H_0 is true, then χ^2 converges in distribution (as $K \to \infty$) to a chi-square distribution with $L! - 1$ degrees of freedom. Thus, for large K, a test with approximate level α is obtained by rejecting H_0 if $\chi^2 > \chi^2_{L!-1,1-\alpha}$, where $\chi^2_{L!-1,1-\alpha}$ is the upper $1-\alpha$ critical point for the chi-square distribution with $L! - 1$ degrees of freedom [Law & Kelton, 2000]. Notice that since this test is based on distributions, it could happen that a deterministic map has no forbidden L-patterns, thus $\nu_j \ne 0$ for all j, however, the null hypothesis be rejected because the ν_j's are not evenly distributed.

4. The Benchmark

For benchmarking we have selected the BDS test [Brock *et al.*, 1996; Sprott, 2003; Liu *et al.*, 1992] for independence in a time series, which is based on the correlation dimension. Since we have used the algorithm provided in [LeBaron, 1997], we will follow this reference for the basics of the BDS test.

Let X_t, $t \ge 1$, be i.i.d. random variables, and

$$I_\epsilon(x, y) = \begin{cases} 1 & \text{if } |x-y| < \epsilon, \\ 0 & \text{otherwise.} \end{cases}$$

The probability that two length-m vectors are within ϵ can be estimated by the correlation sum

$$C_{m,n}(\epsilon) = \frac{2}{n(n-1)} \sum_{s=1}^{n} \sum_{t=s+1}^{n} \prod_{j=0}^{m-1} I_\epsilon(X_{s-j}, X_{t-j}).$$

It is shown in [Brock *et al.*, 1996] that

$$W_{m,n}(\epsilon) = \sqrt{n}\, \frac{C_{m,n}(\epsilon) - C_{1,n}^m(\epsilon)}{\sigma_{m,n}(\epsilon)},$$

where $\sigma_{m,n}(\epsilon)$ is a complicated normalization (see [LeBaron, 1997]), converges in distribution to a standard normal distribution. A statistically significant nonzero value of $W_{m,n}(\epsilon)$ is evidence for determinism in the univariate time series $\{X_t : t \ge 1\}$.

Note that this method relies on the selection of the parameters m and ϵ. Following the usual procedure [Liu *et al.*, 1992], we take $\epsilon = 0.9^j$ with $j = 0, 1, 2, \ldots$. A combination of m and ϵ is *adequate* if a random time series is accepted as deterministic using this test the number of cases prescribed by the level of the test α.

5. Numerical Simulations

For the underlying deterministic time series we use projections on the first coordinate of orbits generated by the Lorenz and time-delayed Hénon maps. As for the additive noise w_n, we use Gaussian white noise,

$$\langle w_m \cdot w_n \rangle = \sigma^2 \delta_{mn},$$

with different standard deviations σ. Simulations with uniformly distributed noise yields similar results (not shown).

We present two types of results: (i) Plots of the number of forbidden (or, rather, missing) ordinal patterns, and (ii) plots of the distribution of the values of the χ^2 statistic. Although the first ones provide only qualitative information, they can eventually complement the information provided by the second ones, as we will see in the case of the Lorenz map.

(i) Let $N_{\max}$ denote the length of the data sequence under scrutiny, and let $\mathbf{n}(L, N)$ be the number of forbidden L-patterns in the initial segment $\xi_0^{N-1} = \xi_0, \xi_1, \ldots, \xi_{N-1}$ of variable length $N \le N_{\max}$. For plotting the number of forbidden patterns, we use overlapping sliding windows of widths $4 \le L \le 7$ and

$$5L! \lesssim N \le N_{\max}. \qquad (4)$$

In order to estimate an average number $\langle \mathbf{n}(L, N)\rangle$ in sequences of the form (2) with $0 \le n \le N-1$ and N complying with the condition (4), we generate 100 sequences of length $N_{\max} = 8000$, and normalize the corresponding count of missing L-patterns.

(ii) For the chi-square test of independence based on the distributions of ordinal L-patterns, we use nonoverlapping windows of widths $L = 4, 5$ and levels $\alpha = 0.1, 0.05$. The corresponding critical points $\chi^2_{L!-1,1-\alpha}$ are the following:

$\chi^2_{L!-1,1-\alpha}$	$\alpha = 0.1$	$\alpha = 0.05$
$L = 4$	32.007	35.172
$L = 5$	139.15	145.46

In order for the sample to be statistically significant, it is sufficient to have

$$5L! \lesssim K, \qquad (5)$$

where K is the number of windows. From (3) we have

$$5L! \lesssim \frac{N_{\max}}{L-1},$$

i.e. $N_{\max} \gtrsim 5(L-1)L!$ This being the case, we take $N_{\max} = 1000$ for $L = 4$, and $N_{\max} = 8000$ for $L = 5$. In order to plot the χ^2-value distribution, a sample of 10 000 sequences was used.

The numerical results are the following.

5.1. *The Lorenz map*

The *Lorenz* (three-dimensional symplectic) *map* [Sprott, 2003] is defined as

$$x_{n+1} = x_n y_n - z_n, \quad y_{n+1} = x_n, \quad z_{n+1} = y_n. \qquad (6)$$

It has an attractor with *Kaplan–Yorke dimension* $D_{KY} = 2$ [Sprott, 2003]. Assuming the well-tested Kaplan–Yorke conjecture $D_{KY} = D_1$, where D_1 is the *information dimension*, then the *fractal dimension* D_0 satisfies

$$D_0 \ge D_1 = 2.$$

Figure 1 shows the return map ξ_{n+1} versus ξ_n for a typical orbit of the Lorenz map on its attractor for Gaussian white noise with $\sigma = 0.25$ (SNR $\simeq$ 10 dB). The geometry of the attractor has been completely washed-out by the noise, but the underlying determinism can be still detected because of the different count of forbidden patterns before (Fig. 2) and after (Fig. 3) switching off the deterministic signal. Not only the count of forbidden ordinal patterns is different, but also the decay rate of the false forbidden patterns with N. The different behavior in Fig. 2 of the curve $L = 4$, on one hand, and the curves $L \ge 5$, on the other hand, strongly indicates that the Lorenz map has no true forbidden 4-patterns.

Figure 4 shows the distribution of the statistic χ^2 obtained for 10 000 sequences generated by the

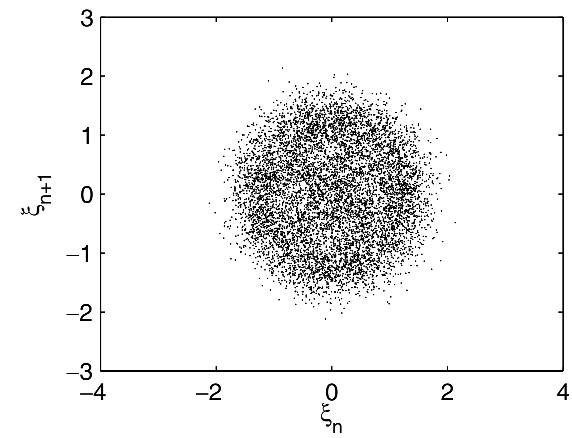


Fig. 1. Return map for a time series of the Lorenz map contaminated with Gaussian white noise with $\sigma = 0.25$ (SNR $\simeq$ 10 dB). The structure of the underlying chaotic attractor has been totally blurred. However, the count of forbidden order patterns is sensibly higher than in the purely random case.

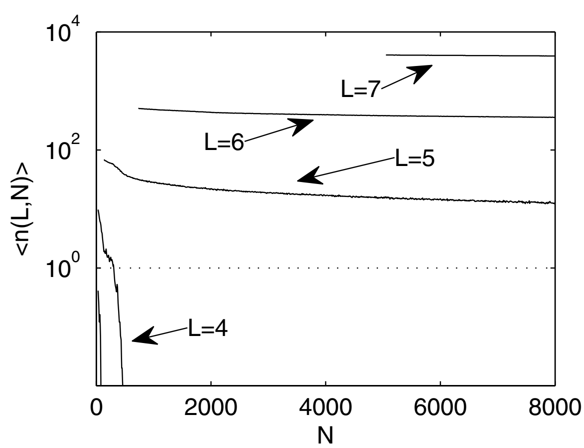


Fig. 2. Average number of forbidden patterns of length L found in a time series of length N, $\langle \mathbf{n}(L, N)\rangle$, (in logarithmic scale) for a noisy series of the Lorenz map with $\sigma = 0.25$ (SNR $\simeq$ 10 dB).

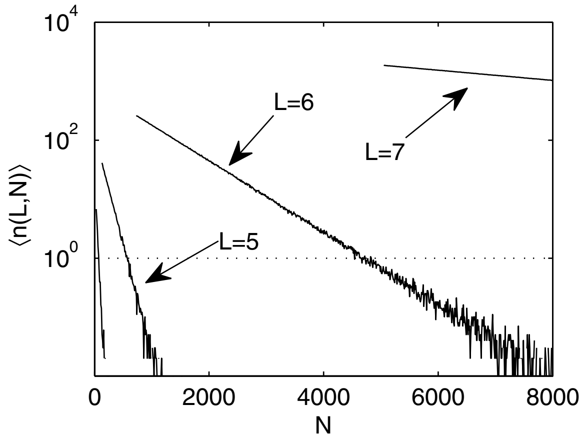


Fig. 3. Average number of forbidden patterns of length L found in a time series of length N, $\langle \mathbf{n}(L, N)\rangle$, (in logarithmic scale) for time series of Gaussian white noise.

Lorenz map contaminated with additive Gaussian noise with $\sigma = 0.25$, 0.50 (SNR $\simeq 10, 4$ dB resp.). In (a) we have used sequences of length $N = 1000$ and nonoverlapping windows of length $L = 4$, while in (b) sequences of length $N = 8000$ and nonoverlapping windows of length $L = 5$ were used. Since the rejection threshold of the null hypothesis H_0 at level $\alpha = 0.05$ is $\chi^2_{23,0.95} = 35.17$ in (a) and $\chi^2_{119,0.95} = 145.46$ in (b), the chi-square test clearly detects determinism. It is worth noticing that the rejection of H_0 in case (a) is due to the nonuniform distribution of the ν_j since, according to Fig. 2, all 4-patterns are visible in noisy time series generated by the Lorenz map with $N \gtrsim 500$ and $\sigma = 0.25$.

Finally, the comparison with the BDS test is depicted in Fig. 5. There, we show the probability P of rejecting the null hypothesis H_0 for the 27 possible adequate BDS tests on a time series of length $N = 1000$ of the Lorenz map, contaminated with Gaussian white noise with $0 \leq \sigma \leq 2$ (thus with SNRs down to approximately -8.9 dB). In the same figure, we have also plotted the probability P of rejecting H_0 using our chi-square test with the same level α. Notice that our test rejects the null hypothesis more often for high noise values ($\sigma \geq 1$), and its performance is comparable to the best one of the BDS test for smaller noise values. We conclude also from Fig. 5 that the BDS test performance strongly depends on the combination of ϵ and m; for some of those combinations, this method wrongly accepts the null hypotheses even for small values of σ.

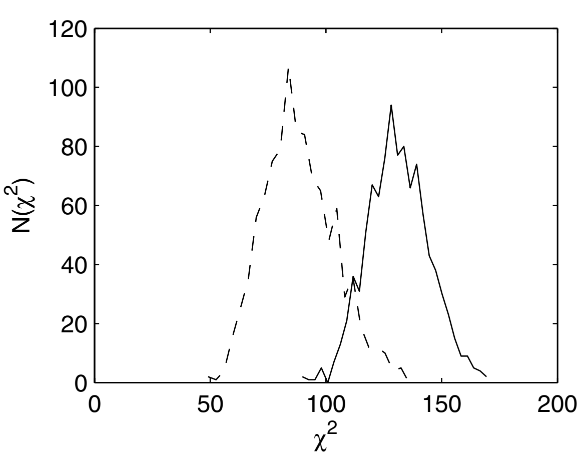
5.2. *The time-delayed Hénon map*

The *time-delayed Hénon map* [Sprott, 2006] is defined as

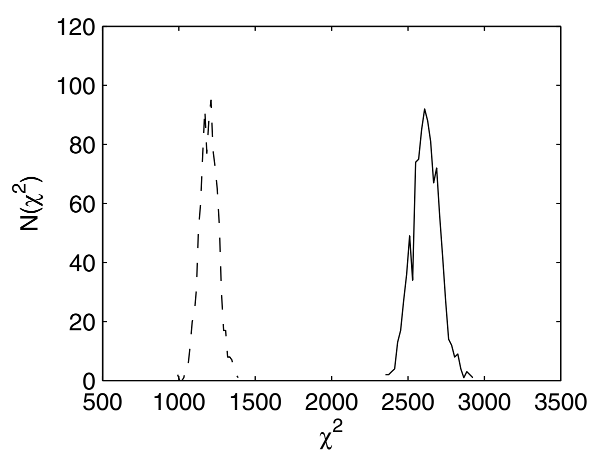
$$x_n = 1 - a x_{n-1}^2 + b x_{n-d}, \tag{7}$$

where a, b are real constants and $d \geq 1$. For $d = 1$, the time-delayed Hénon map is equivalent to the logistic map $x_{n+1} = A x_{n-1}(1 - x_{n-1})$, with [Sprott, 2006]

$$A = \frac{b-1}{2a} \pm \frac{1}{2a}\sqrt{(b-1)^2 + 4a}.$$



(a)



(b)

Fig. 4. Distribution $N(\chi^2)$ of χ^2 for 10 000 noisy sequences generated with the Lorenz map, for $L = 4$, $N = 1000$, $\sigma = 0.25$ (continuous line) and $\sigma = 0.50$ (dashed line) (SNR $\simeq 10, 4.0$ dB resp.) (a), and for $L = 5$, $N = 8000$, $\sigma = 0.25$ (continuous line) and $\sigma = 0.50$ (dashed line) (SNR $\simeq 10, 4.0$ dB resp.) (b).

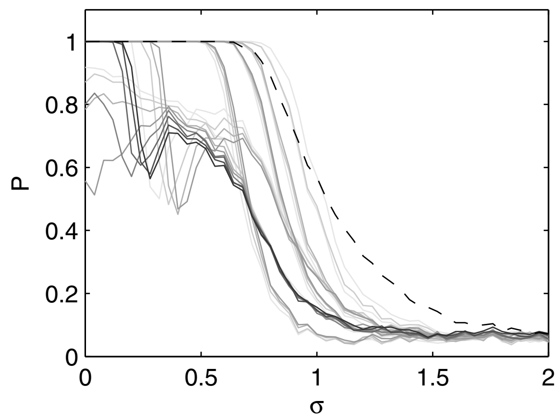


Fig. 5. The continuous lines indicate the probability of rejecting the null hypotheses H_0 (the time series is i.i.d.) for a time series of the Lorenz map contaminated with Gaussian white noise with σ up to $\sigma = 2$ (and thus for SNR down to approximately $-8.9\,\text{dB}$) when applying the BDS test with level $\alpha = 0.05$. In total, 27 tests for different combinations of ϵ and m were performed. The lighter the gray color is, the bigger is the value of ϵ used (see the text for details). The dashed line indicates the probability of rejecting H_0 when using our chi-square test at the same level $\alpha = 0.05$. We can see that our test rejects the null hypotheses more often.

For $d = 2$ and $a = 1.4$, $b = 0.3$, we recover the familiar two-dimensional dissipative Hénon map.

For $a = 1.6$ and $b = 0.1$, Sprott [2006] found the following linear relation between D_{KY} and d over the range $1 \leq d \leq 100$:

$$D_{KY} \cong 0.192d + 0.699.$$

By using again the Kaplan–Yorke conjecture, we infer

$$D_0 \geq D_1 = D_{KY} \cong 0.192d + 0.699$$

for the fractal dimension D_0 of the attractor over the range $1 \leq d \leq 100$. In particular, $D_0 \geq 1.083$ for $d = 2$, $D_0 \geq 10.299$ for $d = 50$ and $D_0 \geq 19.899$ for $d = 100$. Thus, this family of maps can be used to see the performance of the methods described above for different values of D_0.

Figure 6 shows the return map ξ_{n+1} versus ξ_n for a typical orbit on the attractor of the time-delayed Hénon map with $d = 50$, both in the absence of noise [Fig. 6(a)] and with Gaussian white noise with $\sigma = 0.5$ (SNR $\simeq 1.3\,\text{dB}$). Again, the geometry of the attractor has been completely blurred by the presence of this strong noise. However, we can see in Fig. 7 that also in this case, the number of forbidden patterns of length L detected on a time series of length N, $\langle n(L, N)\rangle$, is sensibly larger than in the white noise-only case.

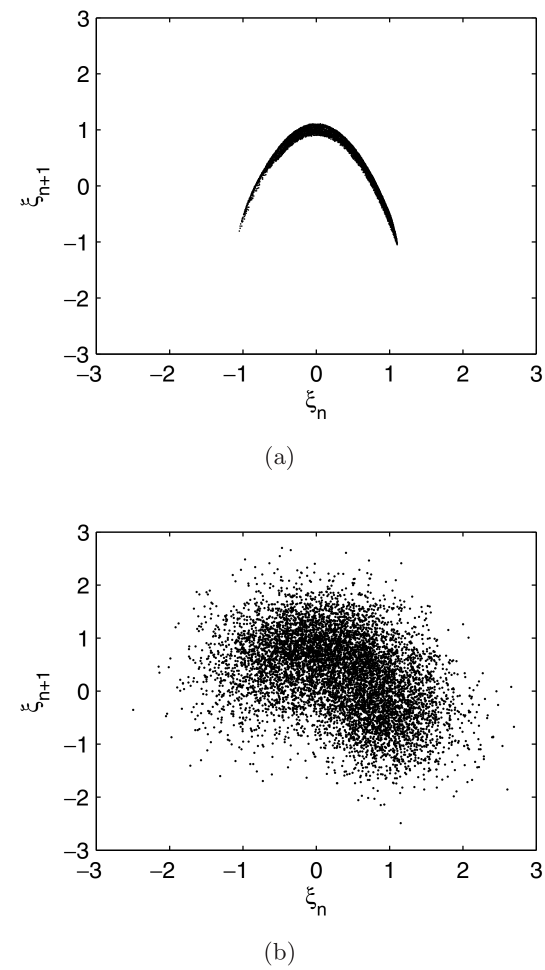


Fig. 6. Return map for a time series of the time-delayed Hénon map with $d = 50$ in the absence of noise (a) and contaminated with Gaussian white noise with $\sigma = 0.5$ (SNR $\simeq$ 1.3 dB) (b). The structure of the underlying chaotic attractor has been totally blurred. However, again here the count of forbidden order patterns is sensibly higher than in the purely random case.

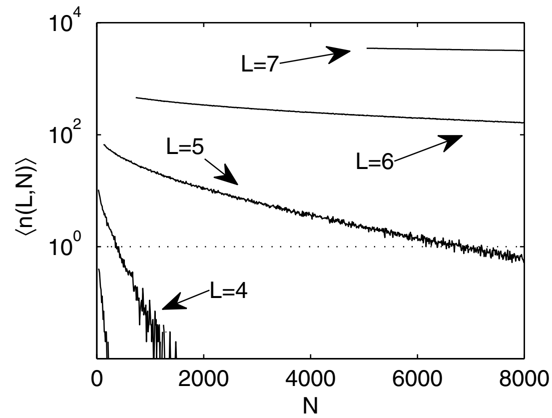
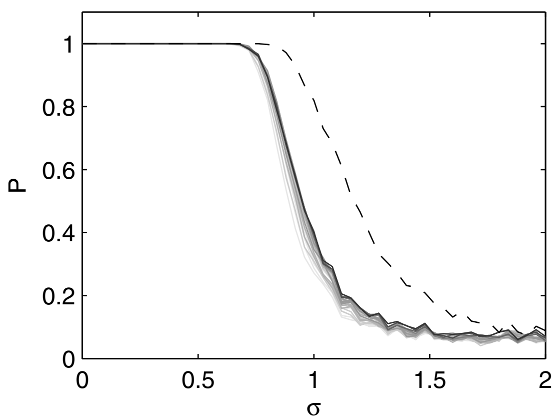
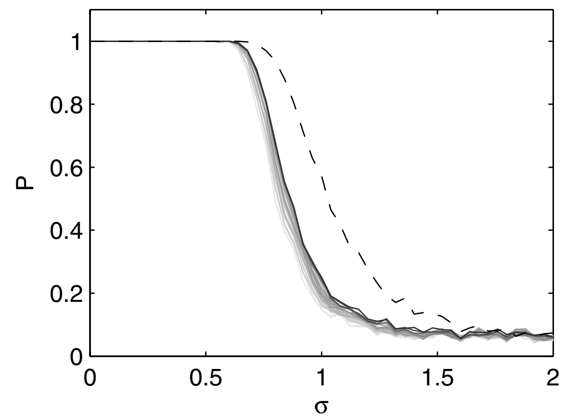


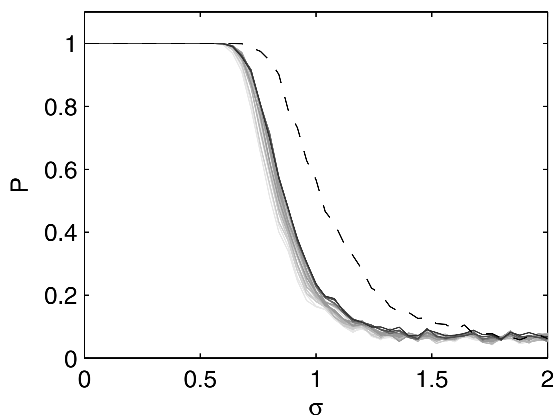
Fig. 7. Average number of forbidden patterns of length L found in a time series of length N, $\langle \mathbf{n}(L, N)\rangle$, (in logarithmic scale) for a noisy series of the time-delayed Hénon map with $\sigma = 0.5$ (SNR $\simeq 1.3\,\text{dB}$).



(a)



(b)



(c)

Fig. 8. Comparison of our method and the BDS test for $d = 2$ (a), $d = 50$ (b) and $d = 100$ (c). Again, the continuous lines indicate the probability of rejecting the null hypotheses H_0 (the time series is i.i.d.) using the BDS test for a time series of the time-delayed Hénon map contaminated with white noise with σ up to $\sigma = 2$ (and thus for SNR down to approximately -10.8 dB for the three cases) when applying the BDS test with level $\alpha = 0.05$. In total, 27 tests for different combinations of ϵ and m were performed. The lighter the gray color is, the bigger is the value of ϵ used (see the text for details). The dashed line indicates the probability of rejecting H_0 when using our chi-square at the same level $\alpha = 0.05$. We can see that our test rejects the null hypotheses more often than the BDS for all noise values and for the three values of d.

Figures 8(a)–8(c) depict the comparison of our method with the BDS test for $d = 2$ and $d = 50$ and $d = 100$, respectively. Again, the probability of a false positive is higher with the BDS test. Since we are interested in the detection of determinism, we may conclude that our method is more reliable.

6. On the Robustness of the Forbidden Patterns

The results above are mainly due to the robustness of the admissible and forbidden ordinal patterns against observational noise. For instance, if a map f has a forbidden pattern of length L, then it is likely that this pattern will also be forbidden in the presence of noise, or that it will appear with a frequency smaller than the frequency expected for white noise. In this section we are going to use a simple example to illustrate the origin of this robustness from a dynamical point of view.

In the following we deal with time series of the form (2), where f is the logistic map $f(x) = 4x(1-x)$ and now w_n is uniform noise in the interval $[-\eta, \eta]$. We know [Amigó *et al.*, 2007] that for $\eta = 0$ this system has one forbidden pattern of length $L = 3$, namely, $\langle 2, 1, 0 \rangle$. In other words, there is no x for which $f^2(x) < f(x) < x$. This can be checked in Fig. 9.

Consider now a time series of this system of length N, $\xi_0^{N-1} = \xi_0, \xi_1, \ldots, \xi_{N-1}$, with its $K = \lfloor N/3 \rfloor$ length-3 windows whose elements do not

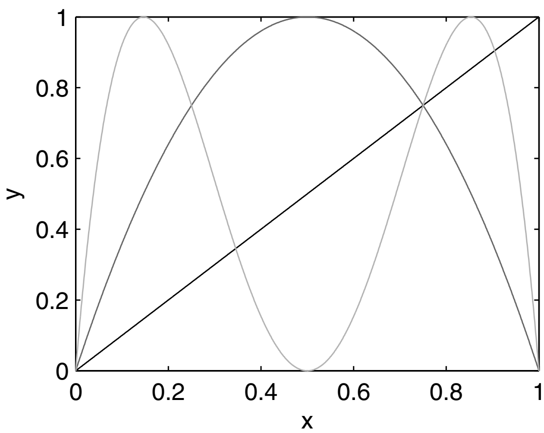


Fig. 9. Plot of the graph of $y = x$ (black), $y = f(x)$ (gray) and $y = f^2(x)$ (light gray) for the logistic map $f(x) = 4x(1 - x)$.

overlap, of the form

$$[\xi_k, \xi_{k+1}, \xi_{k+2}], \quad k = 0, 3, 6, \ldots \tag{8}$$

The question that we want to answer is: if we pick one of these windows randomly, what is the probability of having $\xi_{k+2} < \xi_{k+1} < \xi_k$? We know that $P(\langle 2,1,0\rangle) = 0$ for $\eta = 0$. For $\eta > 0$ it becomes nonzero: it is equal to the probability for

$$x_{k+2} + w_{k+2} < x_{k+1} + w_{k+1} < x_k + w_k \tag{9}$$

to occur.

If η is small, this can only happen if x_k is sufficiently close to any of the two fixed points of the map, $x = 0$ or $x = 3/4$. In that case, $x_k \approx x_{k+1} \approx x_{k+2}$, so noise can make the order relation (9) possible. How close is "sufficiently close" will obviously depend on the value of η. In order to estimate it, assume for example that x_k is close to $x = 0$, and write $x_k = \delta > 0$. Then $x_{k+1} = f'(0)\delta + O_1(\delta^2)$ and $x_{k+2} = (f'(0))^2\delta + O_2(\delta^2)$. Since $\xi_l \in [x_l - \eta, x_l + \eta] \equiv I_l$, Eq. (9) can be fulfilled only if the intervals I_k, I_{k+1} and I_{k+2} overlap. In particular, in order for Eq. (9) to hold, the intervals I_k and I_{k+2} must overlap. By noting that $|O_2(\delta^2)| \leq M_2\delta^2$, where M_2 is estimated using standard techniques (calculating the remainder of the Taylor series), we find that those intervals will surely overlap if

$$\delta \leq \delta_0(\eta) = \frac{1 - f'(0)^2 + \sqrt{(1 - f'(0)^2)^2 + 8M_2\eta}}{2M_2}. \tag{10}$$

In sum, we estimate that x_k is sufficiently close to $x = 0$ (in the sense that Eq. (9) can hold true for η small) if $x_k \in [0, \delta_0(\eta)]$.

We can calculate analogously the $\delta_+(\eta) > 0$ and $\delta_-(\eta) > 0$ such that if $x_k \in [3/4 - \delta_-(\eta), 3/4 + \delta_+(\eta)]$, then x_k is sufficiently close to $x = 3/4$ in the same sense as before.

Thus, the probability $P(\eta)$ for x_k (along with x_{k+1} and x_{k+2}) to lie sufficiently close to any of the fixed points, is equal to the measure μ of those two intervals, that for the logistic map can be calculated explicitly [Alligood *et al.*, 2001], as

$$P(\eta) = \mu\left([0, \delta_0(\eta)] \cup \left[\frac{3}{4} - \delta_-(\eta), \frac{3}{4} + \delta_+(\eta)\right]\right), \tag{11}$$

so that,

$$P(\langle 2,1,0\rangle) \approx P(\eta)P(w_{k+2} < w_{k+1} < w_k) = \frac{P(\eta)}{6}. \tag{12}$$

In order to verify these results, we have calculated numerically the probability P of finding at least once the pattern $\langle 2,1,0\rangle$ in any of the $\lfloor N/3\rfloor$ windows of the form (8) in a time series of N elements. Following our reasoning, this probability should be close to $1 - (1 - P(\eta)/6)^{\lfloor N/3\rfloor}$ for the logistic map contaminated with observational noise of amplitude η, whereas for random time series it should be $1 - (1 - 1/6)^{\lfloor N/3\rfloor}$ [Amigó *et al.*, 2007], which is clearly greater. This is confirmed by Fig. 10. We can see that our estimation of this probability is quite close to the real value, so

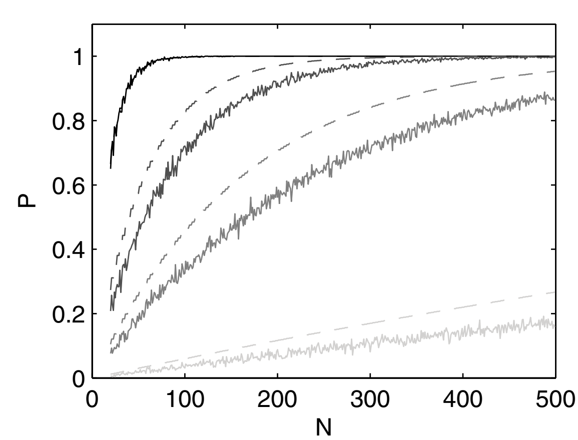


Fig. 10. Numerical (continuous line) computation and analytical estimation (dashed) of the probability P of finding the order pattern $\langle 2,1,0\rangle$ on any of the $\lfloor N/3\rfloor$ windows of the form given by Eq. (8) of a time series of length N, for the logistic map with noise of $\eta = 0.0001$ (light gray), $\eta = 0.01$ (gray), $\eta = 0.1$ (dark gray), and for a random time series (black). Clearly the probability of finding such pattern at least once for noisy time series is smaller than for a random time series.

we think that the ideas described above enlighten the robustness of forbidden patterns against noise. Figure 10 shows that the probability of finding the pattern $\langle 2, 1, 0 \rangle$ on a noisy time series of the logistic map is sensibly smaller than for white noise. This would make a difference when applying a chi-square test like the one described above.

7. Conclusion

We have presented a method to discriminate white noise from deterministic time series corrupted with high levels of white noise. From the comparative study presented in the last section, we infer that the method explained in Sec. 3, based on the properties of ordinal patterns (Sec. 2), compares favorably to the BDS test, one of the standard tests for independence in time series. Furthermore, the BDS algorithm is $O(N^2)$, see [LeBaron, 1997], whereas a simple estimation shows that our chi-square test is approximately $O(N)$. This, together with the fact that our method does not require to adjust parameters like m and ϵ (we just have to investigate the distribution of the ordinal L-patterns satisfying the condition (5)), reinforces the above conclusion.

Acknowledgments

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