Effect of noise on chaotic scattering

Jesús M. Seoane,[1,*] Liang Huang,[2] Miguel A. F. Sanjuán,[1] and Ying-Cheng Lai[2,3]
[1]*Departamento de Física, Universidad Rey Juan Carlos, Tulipán s/n, Móstoles, Madrid 28933, Spain*
[2]*Department of Electrical Engineering, Arizona State University, Tempe, Arizona 85287, USA*
[3]*Department of Physics, Arizona State University, Tempe, Arizona 85287, USA*
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When noise is present in a scattering system, particles tend to escape faster from the scattering region as compared with the noiseless case. For chaotic scattering, noise can render particle-decay exponential, and the decay rate typically increases with the noise intensity. We uncover a scaling law between the exponential decay rate and the noise intensity. The finding is substantiated by a heuristic argument and numerical results from both discrete-time and continuous-time models.

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Chaotic scattering in open Hamiltonian systems has been an area of study in nonlinear dynamics, with applications in a number of fields in physics [1]. Most previous works have been on purely conservative systems [1–4]. More recently, the effects of weak dissipation on chaotic scattering have been addressed [5–7]. Despite a large body of existing literature on chaotic scattering, there have been few works on the effect of noise on characteristics of the scattering dynamics [8–10]. For instance, in Ref. [8], the escapes from a driven potential-well system and the estimation of the average escape time in a noisy environment were addressed. The fact that noise is inevitable in any physical system has motivated us to investigate more thoroughly the interplay between random and deterministic scattering dynamics in a quantitative manner. In this Brief Report we shall report a scaling law relating the particle-decay rate to the noise intensity.

To describe the scaling law, we use the standard setting of classical particle motion in a potential field [3,4]. In such a system, there exists a *scattering region* where interactions between scattering particles and the potential occur, whereas outside the region, the potential is negligible so that the particle motions are essentially free. For many potential functions of physical interest, evolution equations are nonlinear, rendering dynamics chaotic in the scattering region. Since the system is open, this region possesses channels through which particles can enter and/or escape. Due to chaos, particles with slightly different initial conditions can exhibit completely different trajectories in the scattering region before exiting, resulting in dramatically different dwelling times in the region. This situation can be characterized by examining the decay of particles from the scattering region. In particular, imagine that we distribute a large number N_0 of initial particles in the region and examine the number $N(t)$ of particles still present in the region at time t. For hyperbolic chaotic scattering or for nonhyperbolic chaotic scattering under weak dissipation [5–7] or noise [11], the probability for a particle to be in the scattering region $R(t)=\lim_{N_0\to\infty}N(t)/N_0$ decays exponentially with time as follows:

$$R(t) \sim e^{-\gamma t}, \tag{1}$$

where γ is the exponential decay rate. What we have found is that for typical chaotic scattering systems, the rate obeys the following scaling law with the noise intensity ε:

$$\gamma \sim \varepsilon^{\kappa}, \tag{2}$$

where the value of κ depends on the mathematical relationship between the noise intensity ε and the variable of our specific system. We shall provide numerical results from both map and flow models and present a heuristic argument to establish the scaling law.

We consider two numerical models: one a discrete-time map and the other a continuous-time flow. Our map model is given by

$$x_{n+1} = \lambda[x_n - (x_n + y_n)^2/4 - \nu(x_n + y_n)] + u_n,$$
$$y_{n+1} = \lambda^{-1}[y_n + (x_n + y_n)^2/4] + v_n, \tag{3}$$

where $\lambda > 1$ is a system parameter that plays the role of the energy in a realistic case, $\nu \geq 0$ is a dissipation parameter, and u_n and v_n are discrete-time uniform random processes simulating the action of noise. At each time n, the values of u_n and v_n are chosen independently and randomly from uniform probability distribution functions $U(u)$ and $V(v)$ given by

$$U(u) = \begin{cases} \dfrac{1}{2u_0} & \text{if} |u| < u_0, \\ 0 & \text{if} |u| \geq u_0, \end{cases} \tag{4}$$

and

$$V(v) = \begin{cases} \dfrac{1}{2v_0} & \text{if} |v| < v_0, \\ 0 & \text{if} |v| \geq v_0. \end{cases} \tag{5}$$

For convenience we choose $u=v=\epsilon$. When noise is absent, for $\nu=0$ the Hamiltonian-map system was originally introduced in Ref. [12] to study the fractal dimension of nonhyperbolic chaotic scattering. The system was later used in Refs. [5,7] for $\nu \neq 0$ to investigate the effect of weak dissipation on chaotic scattering and the characterization of its fractal dimension. In our computation, the scattering region

*jesus.seoane@urjc.es

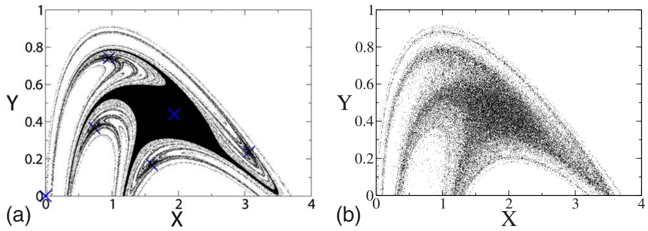

FIG. 1. (Color online) For map model (3) for $\lambda=4$ and $\nu=0.03$, (a) in the absence of noise, basins of attraction of the fixed-point attractor (black) located at the center of the original KAM island and of the scattering trajectories that escape to infinity (blank), and (b) when noise of intensity $\varepsilon=0.2$ is present, initial conditions (black dots) for trajectories that have not exited the scattering region in 100 iterations.

is defined to be $\sqrt{x^2+y^2}<10$. In the noiseless Hamiltonian limit, there is a large Kolmogorov-Arnold-Moser (KAM) island in the phase space. There are a stable fixed point at the center of the island and an unstable fixed point at (0,0). The stable fixed point becomes a fixed-point attractor when weak dissipation is introduced. There is a coexisting attractor at infinity that scattering trajectories approach asymptotically. A typical basin structure of the system in the absence of noise is shown in Fig. 1(a). When noise is present, the fixed-point attractor is destroyed and almost all trajectories escape to infinity. Figure 1(b) shows, for $\varepsilon=0.2$, initial conditions (black dots) for the trajectories that stay in the scattering region for at least 100 iterations. A typical exponential decay law of the number of particles from the scattering region as induced by noise is shown in Fig. 2(a) for $\lambda=4$, $\nu=0.03$, and $\varepsilon=0.2$. As the noise intensity is increased, we observe an increase in γ. Figure 2(b) shows γ versus ε, where the approximately linear behavior indicates scaling law (2) with $\kappa=1$. This value of $\kappa=1$ is due to the linear relationship between the noise intensity ϵ and the variables, as shown in Eq. (3).

We next consider a flow model, the Hénon-Heiles system [13] as defined by the following Hamiltonian: $H=(\dot{x}^2+\dot{y}^2)/2+(x^2+y^2)/2+x^2y-y^3/3$. In the absence of dissipation and noise, two types of motion can occur: bounded and unbounded, depending on the particle energy. Escapes and, consequently, chaotic scattering are possible when the particle energy is above the critical value $E_e=1/6$. In this case,

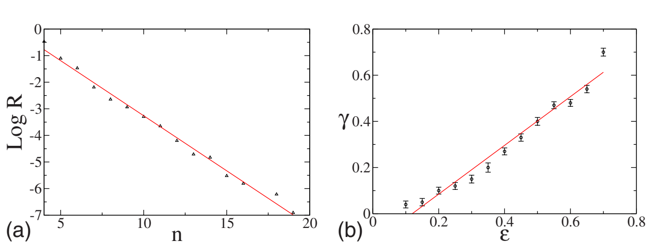

FIG. 2. (Color online) For map model (3) for $\lambda=4$ and $\nu=0$, (a) for $\varepsilon=0.5$, natural logarithm of R versus n, indicating an exponential decay of the probability of particles staying in the scattering region. Initial conditions are chosen from the horizontal line $y_0=-2$ for $x\in(0.5,0.6)$. (b) Exponential decay rate γ versus ε, where we observe the relation $\gamma\sim\varepsilon$.

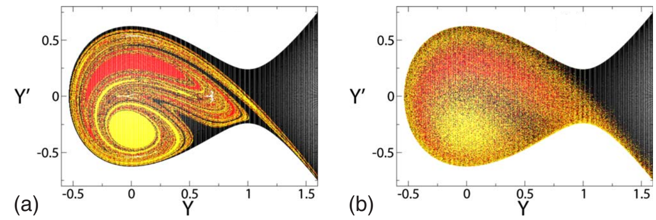

FIG. 3. (Color online) For the Hénon-Heiles system for $E=0.195$, (a) three coexisting exit basins in the conservative limit in the absence of noise, where the colors denote the sets of initial conditions generating trajectory that exit through three different channels, and (b) corresponding plot when noise of intensity $\varepsilon=2\times10^{-3}$ is present.

the system presents a $2\pi/3$ symmetry with three different escaping channels [6,7,13,14]. In the presence of weak dissipation and noise, the equations of motion become

$$\ddot{x}+x+2xy+\mu\dot{x}=\sqrt{2\varepsilon}\xi(t),$$

$$\ddot{y}+y+x^2-y^2+\mu\dot{y}=\sqrt{2\varepsilon}\eta(t), \qquad (6)$$

where μ is a parameter characterizing the amount of dissipation and $\xi(t)$ and $\eta(t)$ are unit Gaussian random processes. We integrate the system by using the standard routine for stochastic differential equations [15]. Typical plots of the exit basin structure of the system in the absence and presence of noise are shown in Figs. 3(a) and 3(b). Note that in the absence of noise [Fig. 3(a)], the three exit basins possess the Wada property [6,14,16]. A typical exponential decay behavior as induced by noise is shown in Fig. 4(a), and the dependence of the decay rate γ on noise intensity is shown in Fig. 4(b). We observe an approximately linear behavior between γ and $\sqrt{\varepsilon}$.

To provide additional numerical support for scaling law (2), we have carried out computations with the Helmholtz oscillator, a dynamical system with escapes [17]. An example of the equation of motion under white Gaussian noise is

$$\ddot{x}+0.1\dot{x}-x-x^2=\sqrt{2\varepsilon}\xi(t). \qquad (7)$$

Following the numerical procedures as for the Hénon-Heiles system, we find that scaling law (2) also holds for the Helmholtz-oscillator system.

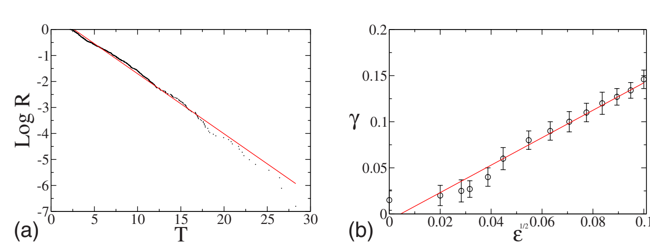

FIG. 4. (Color online) For the Hénon-Heiles system for $E=0.195$ and $\mu\gtrsim0$, (a) for $\varepsilon=0.02$, natural logarithm of R versus T, indicating a typical exponential decay of particles from the scattering region, and (b) dependence of the decay rate γ on the noise intensity ε. We have $\gamma\sim\varepsilon^{1/2}$.

We now provide a heuristic theory to explain scaling law (2). In general, an exponential decay law is the result of a nonattracting Cantor-type fractal set in the phase space. Say we randomly and uniformly distribute a large number N of particles in the scattering region. In the absence of noise, after a unit time interval T_0, η fraction of particles will have escaped. If noise is added at this time, and if the noise is intense enough to effectively redistribute the remaining particles randomly in the initial region, the particle distribution becomes the same as the initial random distribution, with a total number of particles $N(1-\eta)$. After another time T_0, the fraction of particles that escaped will be the same as the previous one, which is η. This illustrates that the effect of noise is to redistribute particles as randomly (uniformly) as possible at different time steps. As a result, the fraction η of escaped particles during identical time intervals is approximately constant. This enables us to write down the evolution equation of the fraction of remaining particles R as $dR/dt = -\eta R$. With initial condition that $R(0)=1$, the solution is $R=e^{-\eta T}$, which is an exponential decay.

Consider the case in which there is no dissipation. In the absence of noise, the decay law is algebraic [18]. If the noise intensity is sufficiently small, we expect most KAM tori to persist. In this case, the decay law is still algebraic. When the noise intensity exceeds a critical value so that KAM tori are smeared out, the decay law will become exponential. Depending on the structures of KAM tori of the specific dynamical system, the transition from algebraic to exponential decay may or may not be abrupt. For example, if KAM tori have a fine structure of typical scale l, then there exists a critical noise intensity $\varepsilon_c \sim l$, where if $\varepsilon < \varepsilon_c$, KAM tori are preserved, making algebraic the scaling law. If $\varepsilon > \varepsilon_c$, KAM tori are destroyed so that the particles are redistributed and the scaling law becomes exponential. However, for most cases, KAM tori may not have a typical scale (self-similar structure). Thus for a given noise intensity, some particles are redistributed effectively and escape in an exponential manner, while others can still be confined for a long time. For this case, since the extremely long time confinement is mainly caused by the fine structures of KAM tori, a small amount of noise will first destroy this confinement. Therefore, the transition from algebraic decay to exponential decay is expected to be smooth in this case.

Consider Eq. (6). Letting $v=\dot{x}$ be the velocity, the first equation can be rewritten as

$$\dot{v} + \mu v - F = \sqrt{2\varepsilon}\xi,$$

where $F=-x-2xy$ is the corresponding force from the Hénon-Heiles potential. Now assume $F=0$. The above equation then becomes a linear stochastic equation describing damped Brownian motion. Regarding v as output and $\sqrt{2\varepsilon}\xi$ as the input, the transfer function is $H(s)=1/(s+\mu)$ and the power spectral density (PSD) satisfies $S_v(f) = |H(2\pi i f)|^2 S_{\sqrt{2\varepsilon}\xi}$. Since ξ is the unit white Gaussian noise, $S_{\sqrt{2\varepsilon}\xi}=2\varepsilon$. Therefore, the variance of the output v is

$$\langle v^2\rangle_T = \int_{-\infty}^{\infty} S_v df = \int_{-\infty}^{\infty} |H(2\pi i f)|^2 [S_{\sqrt{2\varepsilon}\xi}] df = \frac{\varepsilon}{\mu} \sim \varepsilon. \quad (8)$$

Thus the energy fluctuation is proportional to ε. However, for nonzero force F, as in our case, the velocity v and the energy E have nonzero mean values, and $E \sim v^2$. Thus $\delta E \sim v\,\delta v$. Since the force due to the noise is proportional to $\sqrt{\varepsilon}$, for a small time interval Δt the velocity fluctuation is on the order of $\sqrt{\varepsilon}\Delta t$. Since the velocity variation due to the force is much smaller, in a short time interval the velocity can be deemed as a constant; thus, $\delta E \sim v\sqrt{\varepsilon}\Delta t \sim \sqrt{\varepsilon}$ [19]. In general, we can write $\gamma = f(E)$ [20]. Under energy fluctuation δE, we have $\gamma \approx f(E) + f'(E)\delta E$. Since particles escape more quickly if they have higher energies, we see that $f'(E)$ is positive. Due to the energy scaling $\delta E \sim \sqrt{\varepsilon}$, we see that for fixed E, γ depends linearly on $\sqrt{\varepsilon}$ and thereby scaling law (2). A similar argument can be made for maps. For example, in Eq. (3), the noise terms u_n and v_n are proportional to x_{n+1} and y_{n+1}, leading to a linear relation between γ and ε.

In conclusion, by using prototype map and flow models for chaotic scattering, we have demonstrated that particle decay is typically exponential in the presence of noise. We have uncovered a scaling law relating the exponential decay rate to the noise intensity. Our result may help provide insights into realistic problems such as the advection of inertial particles in open chaotic flows or the transport and trapping of chemically or biologically active particles in large-scale flows, where noise is inevitable.

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