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Exponential decay and scaling laws in noisy chaotic scattering

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Abstract

In this Letter we present a numerical study of the effect of noise on a chaotic scattering problem in open Hamiltonian systems. We use the second order Heun method for stochastic differential equations in order to integrate the equations of motion of a two-dimensional flow with additive white Gaussian noise. We use as a prototype model the paradigmatic Hénon–Heiles Hamiltonian with weak dissipation which is a well-known example of a system with escapes. We study the behavior of the scattering particles in the scattering region, finding an abrupt change of the decay law from algebraic to exponential due to the effects of noise. Moreover, we find a linear scaling law between the coefficient of the exponential law and the intensity of noise. These results are of a general nature in the sense that the same behavior appears when we choose as a model a two-dimensional discrete map with uniform noise (bounded in a particular interval and zero otherwise), showing the validity of the algorithm used. We believe the results of this work be useful for a better understanding of chaotic scattering in more realistic situations, where noise is presented.

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1. Introduction

Chaotic scattering in open Hamiltonian systems has been the focus of study during the last decades, with applications in several fields of physics. Previous work has paid attention on purely conservative systems [1–4]. Later work has considered the effects of weak dissipation on these systems [5,6]. Our goal in this Letter refers to the effects of the noise in chaotic scattering problems.

The context of this problem takes place in the motion of a particle in a potential field [2,3]. In general, there exists a region where interactions between scattering particles and the potential occur, whereas outside the region, the potential is negligible so that the particle motions are essentially free. This region is typically called the *scattering region*. For many potential functions of physical interest, evolution equations are nonlinear resulting in chaotic dynamics in the scattering region. Since the system is open, this region possesses channels for which the particles may enter or escape. From the chaotic dynamics in the scattering region, particles with slightly different initial conditions can experience different paths in the region and they can spend different intervals of time in the region and may escape taking different directions.

The lifetime for typical particles coming into the scattering region is finite, exhibiting chaotic dynamics. In this sense chaotic scattering could be presented as a physical manifestation of transient chaos [7,8].

The presence of external perturbations, namely dissipation (Refs. [5,6]), occurring in realistic situations appears in several physical contexts [9]. Also, the presence of noise in open flows has been of interest in recent works [10,11] in order to study superpersistent chaotic transients. The study of the effects of noise on a classical chaotic scattering is not that common and has not been studied in much detail. The influence of noise on a chaotic scatterer has been recently analyzed in the context of the Gaspard–Rice system in the presence of white noise [12]. Our goal here is to analyze the role of weak noise in the dynamics of an open Hamiltonian system.

For numerical integration of the equations of motion in Hamiltonian systems, where the energy is conserved, a symplectic integrator is used. A typical symplectic algorithm com-

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monly used is the second order *leap-frog* method. For systems with dissipation in which the energy is not preserved the most characteristic method is the fourth order *Runge–Kutta*. All these methods are well known in the literature and can be found in Refs. [13,14]. These algorithms work for deterministic systems.

For stochastic processes however, one of the most accurate algorithm is the second order *Heun algorithm* [15]. This is not a symplectic algorithm and cannot be used in systems where the energy is conserved. Since it is not valid for "stochastic Hamiltonian systems",[1] we will introduce a tiny amount of dissipation in the equations of motion to avoid problems with the integration algorithm. This is the reason why we use the cited algorithm. In order to show the efficiency of this algorithm we will also study a two-dimensional discrete model with uniform bounded noise, for which we obtain the same results as in our continuous time example.

This Letter is organized as follows: Section 2 describes the dynamics of a noisy scattering problem. In Section 3 we study the influence of the intensity of noise on the survival probability of the scattering particles in the scattering region. Section 4 provides numerical simulations with a discrete map to show the validity of the algorithm used. Conclusions and discussions are presented in Section 5.

2. Dynamics in noisy chaotic scattering

In this section we will study the dynamics of chaotic scattering in a noisy environment. In order to show the dynamics of this kind of systems we will use the Hénon–Heiles system with weak dissipation in Ref. [6].

The Hénon–Heiles system is described by the Hamiltonian

$$H = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}\left(x^2 + y^2\right) + x^2 y - \frac{1}{3}y^3, \tag{1}$$

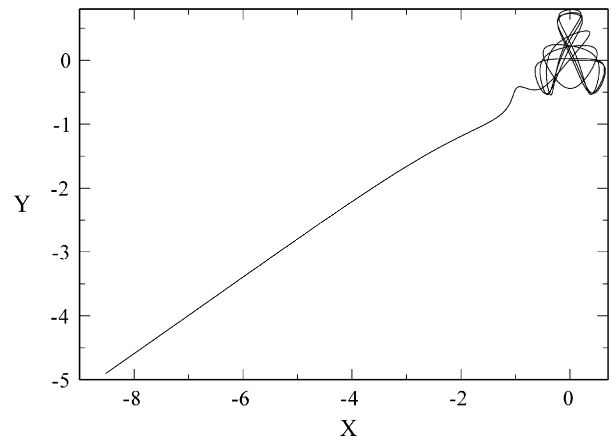
which defines the motion of a unit mass particle in the two-dimensional potential

$$V(x, y) = \frac{1}{2}\left(x^2 + y^2\right) + x^2 y - \frac{1}{3}y^3. \tag{2}$$

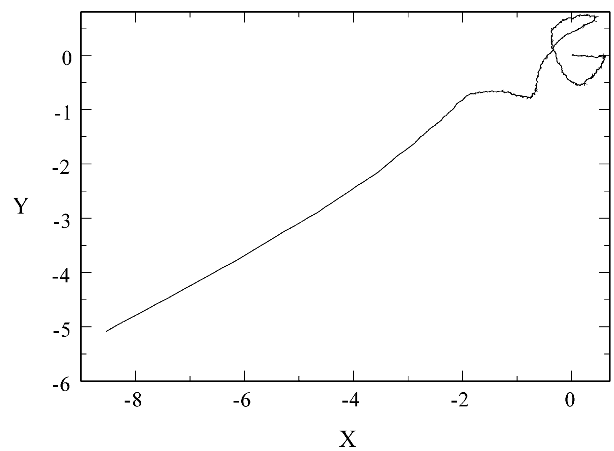
The system was originally presented in 1964 [16] and it is a typical system with escapes in dynamical astronomy. Two main types of motion can be distinguished in this potential, which correspond to bounded and unbounded orbits. According to the value of the energy, the orbit is trapped in a region or escapes from it. Specifically, this threshold value of the energy for which the particle can escape to the infinity is called the *escape energy*, E_e, and this value is $E_e = 1/6 = 0.1666$. Escapes are possible for values of energy above the escape energy and the motions are unbounded. In this situation the system presents triangular symmetry with three different channels for which the particles may escape (see, for example, Refs. [6,16,17]).

When we introduce a linear dissipation term in the system, proportional to the velocity, as in Ref. [6], and Gaussian noise,

[1] The second order Heun algorithm is not a symplectic algorithm. It means that, in principle, it is no so longer valid for conservative (Hamiltonian) systems. This is the reason for which a tiny small of dissipation is introduced in the system.



(a)



(b)

Fig. 1. These figures show single trajectories of the Hénon–Heiles system with weak dissipation (described in Eq. (3)). The parameters values are $E = 0.19$ and $\mu \gtrsim 0$, where the initial condition is at the point $(x_0, y_0) = (0, 0)$, with $\theta_0 = 0$. (a) In the absence of noise, $\epsilon = 0$. (b) When a small amount of noise is introduced, $\epsilon = 10^{-3}$.

the equations of motion read

$$\begin{aligned} &\ddot{x} + x + 2xy + \alpha\dot{x} = \sqrt{2\epsilon}\,\xi(t), \\ &\ddot{y} + y + x^2 - y^2 + \beta\dot{y} = \sqrt{2\epsilon}\,\eta(t), \end{aligned} \tag{3}$$

with α and β being dissipative parameters, ϵ the intensity of the noise and $\xi(t)$ and $\eta(t)$ random variables. For convenience, we fix $\alpha = \beta = \mu$, where μ is the dissipative parameter. Since we are working with stochastic differential equations we will use the numerical integration explained in Ref. [15].

In Figs. 1(a) and 1(b) typical trajectories of this system for parameters values $E = 0.19$, $\mu \simeq 0$, and $\epsilon = 0$ (Fig. 1(a)) and $\epsilon = 10^{-3}$ (Fig. 1(b)), are shown. For convenience, we launch particles from within the scattering region and examine the escaping trajectories. Specifically, the particles are shot from the vertical line segment centered at $(x_0, y_0) = (0, 0)$ with angle with respect to the x-axis $\theta_0 = 0$. The addition of noise results in a classical trajectory in a noisy environment (similar to a random walk) causing particles to typically to escape earlier than in the noiseless case.

The phase space in open Hamiltonian systems with three or more exits has a very rich fractal structure. In the noise-

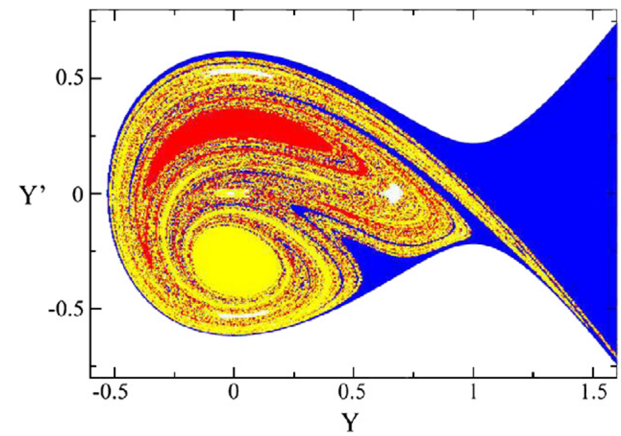


Fig. 2. Exit basins for the noiseless Hénon–Heiles Hamiltonian, Eq. (1), where Wada basins are shown. The colors (yellow, red and blue) denote the set of initial conditions of particles that are escaping through one of the three different exits that the system presents. Here, we take $E = 0.19$ and $Y' = \dot{Y}$. (For interpretation of the references to colour in this figure legend, the reader in referred to the web version of this Letter.)

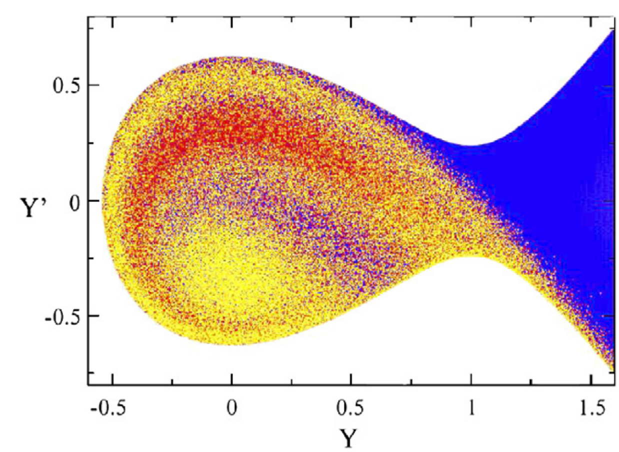


Fig. 3. Exit basins of the Hénon–Heiles system with weak dissipation and parameters $E = 0.19$, $\epsilon = 10^{-3}$ and $Y' = \dot{Y}$. The phase space appears smeared because of the effect of the intensity of noise, ϵ, and Wada basins disappear. The colors (yellow, red and blue) indicate different exits for the shot particles where each trajectory is computed for a different random realization of the noise process. (For interpretation of the references to colour in this figure legend, the reader in referred to the web version of this Letter.)

less case our system also has a strong topological property, the Wada property. This topological property has been shown in several dynamical systems, the Hénon–Heiles system [17] and the weak dissipative Hénon–Heiles system [6], among others. Other references where Wada basins can be found are [18–23].

In noiseless dynamical systems the concept of a basin is very important. For dissipative dynamical systems where attractors are common, we define a basin of attraction as the set of initial conditions that leads to an attractor. In conservative dynamical systems with escapes where the attractors do not exist, we define an exit basin in the same way as for a basin of attraction in dissipative systems. An exit basin is the set of initial conditions that leads to a certain exit (typically to infinity).

A basin B satisfied the Wada property if any initial condition which is on the boundary of one basin is also simultaneously on the boundary of, at least, two other basins.

A plot of the basins of attraction of this system is shown in Fig. 2, for parameters values $E = 0.19$ and $x_0 = 0$, and $\mu = 0$. This figure shows Wada basins as is proved in Ref. [17].

As is well known, the addition of weak dissipation does not affect the structural stability of the Wada basins [6]. On the contrary, if we introduce a small amount of noise, for instance $\epsilon = 10^{-3}$, the topology of the Wada basins is destroyed and the phase space is smeared as shown in Fig. 3.

3. Decay law and scaling in noisy chaotic scattering

In this section we will study the behavior of the scattering particles in the scattering region when a small amount of noise is introduced in the system. We are interested in the dependence on time of the survival probability of the particles in the scattering region.

In Hamiltonian systems regular motions are very typical and fundamental. In particular we are referring to motions on Kolmogorov–Arnold–Moser (KAM) tori [24,25]. In hyperbolic chaotic scattering, all the periodic orbits are unstable and there are no KAM tori in the phase space. In this case, the particle decay law is exponential. On the contrary, in nonhyperbolic chaotic scattering, KAM tori coexist with chaotic saddles, which typically results in algebraic decay in the survival probability of a particle in the scattering region [25].

Here we will focus on how noise affects the nonhyperbolic regime. Previous works [5,6] studied the effect of weak dissipation in the survival probability of the particles in a nonhyperbolic regime. The authors, in Refs. [5,6], found that the algebraic decay law is structurally unstable in the sense that it immediately becomes exponential in presence of some amount of dissipation, no matter how small it is. This fact is due to the appearance of new attractors when a small amount of dissipation is introduced into the system (all details of this phenomenon can be found in Ref. [5]). This result is consistent with the fact that hyperbolic dynamics in Hamiltonian systems are typically structurally stable while nonhyperbolic dynamics are not. The fraction of particles remaining in the scattering region due to the effects of dissipation depends of the dissipative parameter μ. For instance, if $\mu = 0.01$, the fraction of particles remaining in the scattering region due to the effects of the attractors, f, is approximately $f \simeq 65\%$.

Recall that the escape time is the time that a particle spends in the scattering region before escaping from it. In the Hénon–Heiles system these escapes take place when the particles cross another orbit, the unstable orbit called *Lyapunov orbit*, since this violates uniqueness of the solution. Once this happens the particles never return to the scattering region and subsequently escape to infinity [6,16,17,26].

The effects of noise in the survival probability of the particles in a nonhyperbolic regime has not been the attention focus in this topic in the previous mentioned works. The fraction of particles remaining in the scattering region, R, versus the escape time, T, where we shoot 500 particles with energy

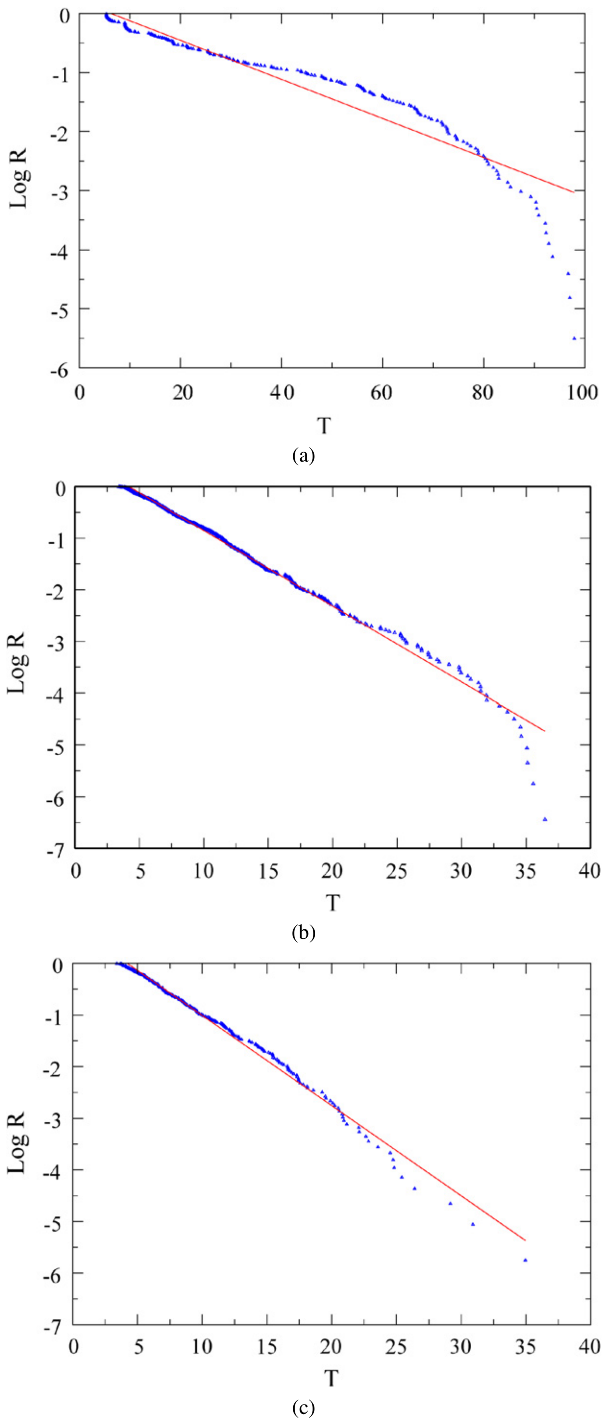


Fig. 4. (a) and (b) represent the decay law for the particles remaining in the scattering region. R denotes the fraction of particles leaving from the scattering region. In (a) and (b) we observe the abruptly change from algebraic to exponential due to the effect of noise. Here, we are shooting 500 particles with energy $E = 0.19$ from $(x_0, y_0) = (0, 1)$ and $\theta \in (0, 2\pi)$. Other parameters are $\mu \geqslant 0$ and $\epsilon = 0.01$. (c) represents the exponential behavior shown in (b) in absence of dissipation.

$E = 0.19$ from $(x_0, y_0) = (0, 1)$ and $\theta \in (0, 2\pi)$ is plotted in Fig. 4, where the parameters values are $\mu \gtrsim 0$ and $\epsilon = 0.01$. In this log plot we easily observe the exponential behavior of the decay law, $R(t) \sim e^{-\gamma t}$ (γ is the coefficient of the decay law commonly called characteristic time), where for this specific

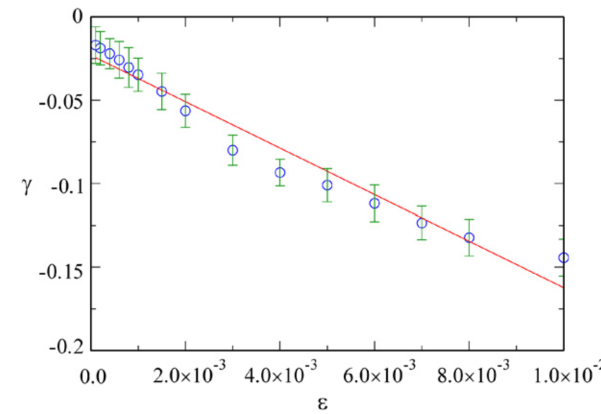


Fig. 5. In this figure we plot the coefficient of the exponential law, γ, of the scattering particles against the noise intensity, ϵ. We choose, as parameters, $E = 0.19$ and $\mu \simeq 0$. We approximately observe a linear scaling relationship between γ and ϵ in the interval of ϵ used where the error bars have at 95% confidence level.

case $\gamma \simeq -0.15$. The drastic change in the decay law from algebraic (see Fig. 4(a)) to exponential is due to the fact that noise destroys the KAM islands and also the attractors of the system and helps the particles to escape from the scattering region earlier than in the case without noise, transforming the algebraic law into an exponential law. Since we introduce a very small amount of dissipation in the system to avoid problems in the integration algorithm, we also use another algorithm for stochastic differential equations in order to show that this result is due to the effects of noise independently of the presence of dissipation. By using the *Euler–Maruyama* algorithm (see Refs. [15,27]) we obtain in Fig. 4(c), in absence of dissipation, the exponential behavior obtained in Fig. 4(b), where a very small amount of dissipation is present.

Another important point regarding the effect of noise on the scaling laws between the characteristic parameters of the system. If we plot the coefficient of the exponential γ versus the intensity of noise ϵ we approximately find a linear scaling relationship between them in the interval used in the numerical simulations. Fig. 5 shows the linear scaling where the correlation coefficient of the fitting is $r = 0.987$. We have fixed $E = 0.19$ and $\mu \simeq 0$ as parameters, where the error bars have the 95% confidence level. This last result is very important in the sense that we can predict (for one specific system) the evolution of the particles in the scattering region if we can control the parameters of the system.

4. Numerical simulations for a two-dimensional discrete map

We will now provide more numerical arguments for the results obtained in Sections 2 and 3 using a discrete dynamical system with noise. We shall use a two-dimensional map used by Motter et al. (Ref. [5]) introducing a uniform noise in it.

This map reads

$$x_{n+1} = \lambda\left[x_n - (x_n + y_n)^2/4 - \nu(x_n + y_n)\right] + u_n,$$
$$y_{n+1} = \lambda^{-1}\left[y_n + (x_n + y_n)^2/4\right] + v_n, \tag{4}$$

where $\lambda > 1$ and $\nu \geqslant 0$ are the parameters of the system and u_n and v_n are noise. At each time n, the values of u_n and v_n are chosen independently and randomly from uniform probability distribution functions $U(u)$ and $V(v)$ given by

$$U(u) = \begin{cases} \frac{1}{2u_0} & \text{if } |u| < u_0, \\ 0 & \text{if } |u| \geqslant u_0, \end{cases} \tag{5}$$

$$V(v) = \begin{cases} \frac{1}{2v_0} & \text{if } |v| < v_0, \\ 0 & \text{if } |v| \geqslant v_0. \end{cases} \tag{6}$$

For convenience we will take $u_0 = v_0 = \epsilon$. The other parameters of the systems have the following physical meanings: the parameter λ plays the same role as the energy in a real system and ν plays the role of the dissipation with the map becoming conservative if $\nu = 0$ and dissipative if $\nu > 0$. The dynamics of this system becoming nonhyperbolic for $\lambda \leqslant 6.5$ and hyperbolic for $\lambda \geqslant 6.5$. We mostly study this map in the nonhyperbolic regime where dissipation is $\nu \geqslant 0$ since we are interested in the effect of noise on the system.

In this system we define the scattering region as the region of points remaining in $x^2 + y^2 < r$, where we take $r = 100$. After an orbit is outside of this area we consider that it has escaped.

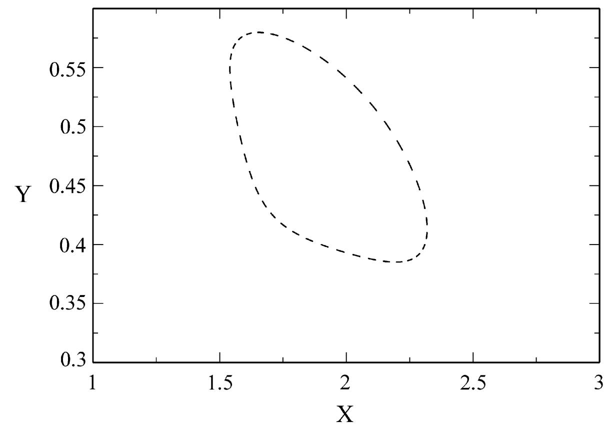
In Figs. 6(a) and 6(b), single trajectories for $\nu = 0$ and different values of noise intensity are shown. We can see that noise destroys the KAM island (in the case of the appearance of attractors the noise destroys them) helping the particle to reach the exit, as shown in Fig. 6(b).

This system presents one stable fixed point at the center of the island and another unstable fixed point at $(0, 0)$. The stable fixed point becomes a fixed point attractor when dissipation is introduced. The typical basin of attraction of this system is plotted in Fig. 7, where the parameters are $\lambda = 4$ and $\nu = 0.01$. If we add a small amount of noise the basin is destroyed (as expected) and smearing appears. After several iterations all orbits escape as shown in Fig. 8. Here, the basin of attraction is plotted after 1000 iterations where the intensity of noise is $\epsilon = 0.05$.

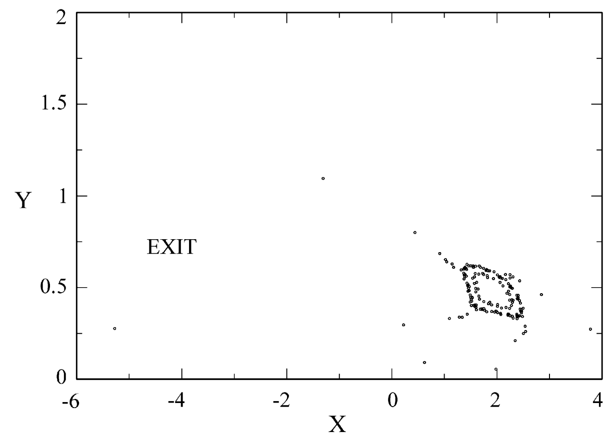
If we now study the effects of the noise in the survival probability of the particles scattered in the scattering region, we find an exponential relationship between the fraction of particles remaining in the scattering region and the escape time, $R(t) \sim e^{-\gamma t}$, as in the case of the flow with additive white Gaussian noise. In Fig. 9 we observe the exponential behavior due to the effects of the noise. For convenience, we choose initial conditions from the horizontal line $y_0 = -2$ and $x \in (0.5, 0.6)$ with parameters values $\lambda = 4$, $\nu = 0$ and $\epsilon = 0.4$. In this case, the value of the coefficient of the exponential law is $\gamma \simeq 0.25$.

If we now plot the coefficient of the exponential law in the survival probability of the particles in the scattering region, γ, against the intensity of noise, ϵ, we obtain a similar scaling law as in the case of the flow as shown in Fig. 10 for the interval of ϵ used in the simulations. The error bars estimated are at the 95% confidence level and the correlation coefficient of the fitting is $r = 0.993$.

These results are consistent with our numerical simulations made in the previous sections.



(a)



(b)

Fig. 6. The figures show different trajectories of the two-dimensional map for parameters values $\lambda = 4$ and $\nu = 0$, where the initial condition is at the center of the island at the point $(x_0, y_0) = (1.9, 0.4)$. (a) We can observe the KAM island for $\epsilon = 0$. (b) For $\epsilon = 0.05$ the KAM island is destroyed and the particle is escaping after several iterations.

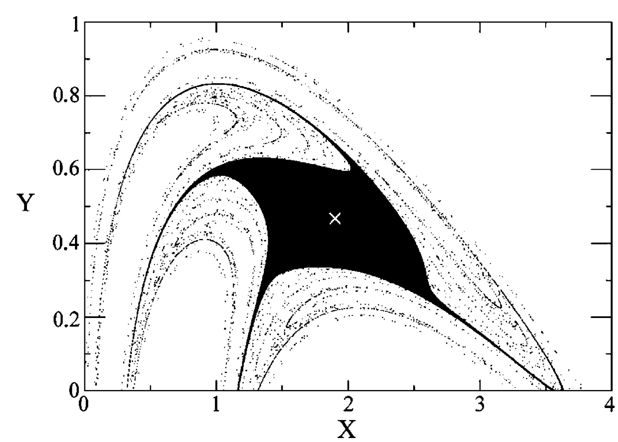


Fig. 7. Basin of attraction (denoted in black) of the fixed-point attractor located in the center of the island (marked by the white cross) of the map for parameter values $\lambda = 4$, $\nu = 0.01$ and $\epsilon = 0$. This figure has been adapted from a figure in Ref. [5].

5. Conclusions and discussions

In conclusion, by means of the second order *Heun algorithm* for stochastic differential equations we have studied the effects

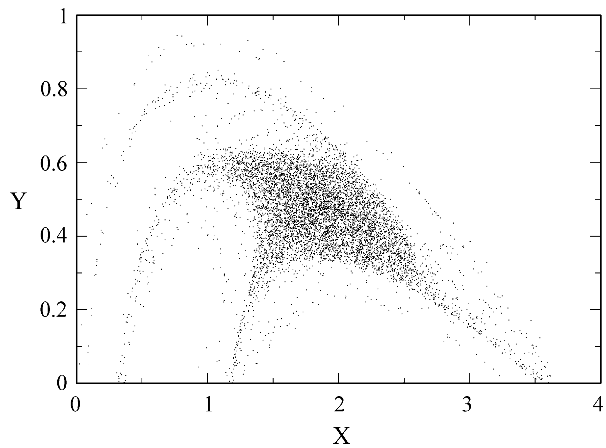


Fig. 8. Basin of attraction of the map (denoted by black points) for $\lambda = 4$, $\nu = 0.01$ and $\epsilon = 0.05$, after 1000 iterations. The fixed-point attractor is destroyed after introducing noise, and all orbits escape after a sufficiently long time period.

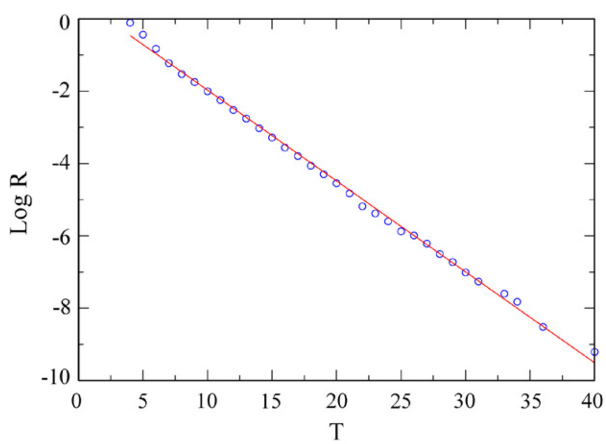


Fig. 9. This figure represents, for the map, the exponential law for the particles remaining in the scattering region. R denotes the fraction of particles leaving from the scattering region. In this figure we observe an exponential relationship between R and T due to the noise effect. We choose initial conditions from the horizontal line $y_0 = -2$ and $x \in (0.5, 0.6)$ with parameters values $\lambda = 4$, $\nu = 0$ and $\epsilon = 0.4$.

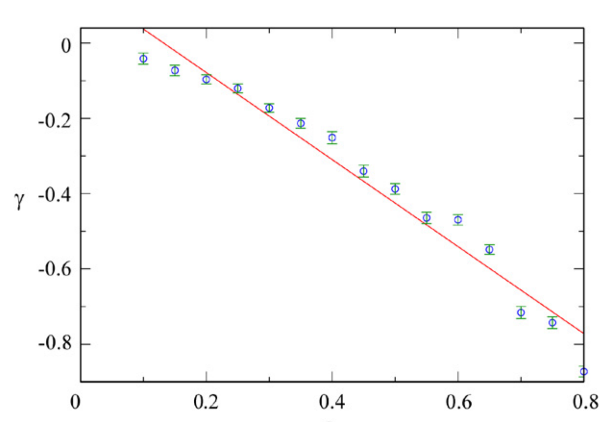


Fig. 10. Plot of the coefficient of the exponential law of the scattering particles against the noise intensity. We choose, as parameters, $\lambda = 4$ and $\nu = 0.01$, throwing the particles from $y_0 = -2$ and $x \in (0.5, 0.6)$. We approximately observe a linear relationship between γ and ϵ with at 95% confidence level in the error bars in the interval of ϵ used in the simulations.

of noise on a classical chaotic scattering problem. By using as a prototype model the Hénon–Heiles system with weak dissipation, we introduce additive white Gaussian noise. We plotted the basins of attraction of the system showing that Wada basins are destroyed after introducing a small amount of noise in the system, hence the phase space appears smearing and its original structure is vanished. Interestingly, we find that the survival probability for the scattering particles in the scattering region changes drastically from algebraic to exponential when the noise is introduced. Moreover, we find a linear scaling law for the coefficient of the exponential law against the intensity of noise. We show the generality of the effects of noise with a two-dimensional map with uniform noise showing that the results are, generally, independent of the chosen system. Since noise appears in all realistic physical situations, our work provides new results and behaviors in the classical chaotic scattering problems.

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